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Nonlinear response of shear deformable beams on tensionless nonlinear viscoelastic foundation under moving loads

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A B S T R A C T

In this paper, a boundary element method is developed for the geometrically nonlinear response of shear deformable beams of simply or multiply connected constant cross-section, traversed by moving loads, resting on tensionless nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections under general boundary conditions. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse moving loading as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Three boundary value problems are formulated with respect to the transverse displacement, to the axial displacement and to a stress functions and solved using the Analog Equation Method, a Boundary Element based method. Application of the boundary element technique yields a system of nonlinear Differential–Algebraic Equations, which is solved using an efficient time discretization scheme, from which the transverse and axial displacements are computed. The evaluation of the shear deformation coefficient is accomplished from the aforementioned stress function using only boundary integration. Analyses are performed to illustrate, wherever possible, the accuracy of the developed method, to investigate the effects of various parameters, such as the load velocity, load frequency, shear deformation, foundation nonlinearity, damping, on the beam displacements and stress resultants and to examine how the consideration of shear and axial compression affects the response of the system.

1. Introduction

Vibration analysis of beams traversed by moving load is of great interest in the area of high-speed transportation or rocket-sledge technology. This problem can be modeled as a beam on elastic foundation subjected to loading moving at a constant speed. According to the modeling of the mechanical behavior of the soil and the soil–foundation interaction, the earliest and probably the simplest mathematical model adopted is the Winkler elastic foundation [1]. In this model the supporting soil behavior is approximated by a series of closely spaced, mutually independent, linear elastic vertical spring elements, providing resistance in direct proportion to the deflection of the beam. However, the application of this model is restricted to non-cohesive soil media due to its inability to take into account the continuity or cohesion of the soil (interaction between adjacent springs). To overcome this weakness, a second parameter is introduced (Filonenko–Borodich,

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Pasternak or Hetenyi models) to account for the interaction among the linear elastic springs [2]. The induction of this second parameter brings the modeling of the soil behavior closer to reality but its response is still not as complicated as the elastic continuum model. This fact resulted in the development of more sophisticated models comprising three independent parameters for the description of the soil behavior. More specifically, since in practice the support structure may be highly nonlinear due to the foundation hardening characteristics (e.g. rail-bed and ballast), the inclusion of a third parameter associated with the cubic nonlinearity of the deflection was verified experimentally by Dahlberg [3]. Moreover, the three-parameter foundation including material damping is a very practical model for dynamic loading cases.

When the beam deforms the aforementioned conventional elastic foundation models can sustain both compression as well as tension. These bilateral foundation models were probably motivated more by the desire of mathematical simplicity than by physical reality. For instance, a railroad rail actually lifts off the ballast in front of the moving train. In order to address this issue, tensionless foundation models were proposed, in which regions of no contact develop beneath the beam. These regions are unknown and the change of the transverse displacement sign provides the condition for the determination of the contact region. Besides, having in mind the magnitude of the arising compressive forces due to environmental loads such as changes in temperature or moisture or due to the train wheel and the importance of weight saving in engineering structures, the study of nonlinear effects on the analysis of supporting structural elements becomes essential. This nonlinearity results from retaining the square of the slope in the strain–displacement relations (intermediate nonlinear theory), avoiding in this way the inaccuracies arising from a linearized second-order analysis. Moreover, due to the intensive use of materials having relatively high shear modulus, the error in the beam analysis incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading. All of the aforementioned concepts constitute the motive for a rigorous nonlinear dynamic analysis of shear deformable beams subjected to moving loads and resting on a tensionless nonlinear elastic foundation.

When the beam deflections are small, a wide range of linear analysis tools, such as modal analysis, can be used, and some analytical results are possible. Analytical solutions of problems involving beam vibrations of simple geometry and boundary conditions under moving loads have received a good amount of attention in the literature, with the pioneer work of Krylov [4] and later the one of Timoshenko [5] who determined dynamic stresses in the beam structure. Linear transverse vibrations of a simply supported beam traversed by a constant force moving at a constant velocity were presented by Inglis [6], Lowan [7] and later on by Koloušek [8] and Fryba [9]. In these approaches the results are usually expressed as an infinite sum of normal modes, obtaining the contribution of each mode by methods of integral transformation. Nevertheless, due to the simplifying assumptions, it must be stressed that the results obtained correspond only to an estimate of the structural response to a moving load.

Since then, important development has also been achieved regarding *linear* more rigorous dynamic analyses of beams under moving loads employing either analytical or numerical methods. To begin with, Weitsman [10] presented an Euler–Bernoulli beam subjected to a concentrated load moving with constant speed resting on a tensionless foundation, relating the load amplitudes that bring the beam to the verge of separation from the foundation to the velocity of motion. Choros and Adams [11] investigated the steady state deformation of an infinite beam on a tensionless undamped elastic foundation under a single moving force. Thambiratnam and Zhuge [12] studied the dynamic analysis of beams on elastic foundation subjected to moving point loads employing the finite element method and modeling the foundation by springs of variable stiffness. Sun [13–15] proposed closed-form displacement responses of beam-type structures subjected to moving line or concentrated loads after obtaining a Green's function of the beam on an elastic or viscoelastic foundation by means of Fourier transform. Kargarnovin and Younesian [16–18] studied the response of infinite beams supported by nonlinear or Pasternak-type viscoelastic foundations subjected to harmonic moving loads employing a perturbation methods. Muscolino and Palmeri [19] scrutinized beams resting on viscoelastically damped foundation under moving sdof oscillators through a state-space formulation, in which a number of internal variables are introduced with the aim of representing the frequency-dependent behavior of the viscoelastic foundation. Zehsaz et al. [20] studied the dynamics of railway, as a Timoshenko beam of limited length, lying on a Pasternak viscoelastic foundation, subjected to moving load employing the modal superposition method. Chen and Chen [21] studied the effect of damping on the multiple steady state deformations of an infinite beam resting on a tensionless Winkler-type foundation subjected to a point load moving with a sub-critical speed. Dimitrovova [22] presented the transverse vibrations induced by a load moving at a constant speed along a finite or an infinite beam resting on a piecewise homogeneous viscoelastic foundation employing the normal-mode analysis and paying attention to the amplification of the vibrations arising from a foundation discontinuity. Ansari et al. [23] studied the vibration of a finite Euler–Bernoulli beam, supported by nonlinear viscoelastic foundation and traversed by a moving load employing the Galerkin method, while the solution for different harmonics is obtained using the Multiple Scales Method.

As the beam deflections become larger, the induced geometric nonlinearities result in effects that are not observed in linear systems. Contrary to the good amount of attention in the literature concerning the linear dynamic analysis of beams supported on elastic foundation, very little work has been done on the corresponding nonlinear problem. Chang and Liu [24] performed the deterministic and random vibration nonlinear analysis of a beam on an elastic foundation subjected to a moving load employing the Galerkin method in conjunction with the finite element method, while the nonlinear system of differential equation has been solved by the implicit direct integration method. Rotary inertia and shear deformations are neglected, while the effects of longitudinal deflections and inertia have been considered so that the coupled equations of longitudinal and transverse deflections can be derived based on Bernoulli–Euler hypothesis. Chen et al. [25] performing a geometrically nonlinear analysis with constant axial force presented the dynamic stiffness matrix of an infinite Timoshenko beam on viscoelastic foundation subjected to a harmonic moving load and determined the critical velocities and the resonant frequencies. Kim and Cho [26] presented the vibration and buckling of an infinite beam-column under constant axial force, resting on an elastic foundation and subjected to moving loads of either constant or harmonically varying amplitude with a constant advance velocity, taking into account shear deformation effect.

In this paper, a Boundary Element Method (BEM) is developed for the geometrically nonlinear response of shear deformable beams of simply or multiply connected constant cross-section, traversed by moving loads, resting on tensionless nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections under general boundary conditions. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse moving loading as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Three boundary value problems are formulated with respect to the transverse displacement, to the axial displacement and to a stress functions and solved using the Analog Equation Method [27], a BEM based method. Application of the boundary element technique yields a system of nonlinear Differential–Algebraic Equations (DAE), which is solved using an efficient time discretization scheme, from which the transverse and axial displacements are computed. The evaluation of the shear deformation coefficient is accomplished from the aforementioned stress function using only boundary integration. The essential features and novel aspects of the present formulation compared with the previous ones are summarized as follows:

(i) Shear deformation effect and rotary inertia are taken into account in the nonlinear dynamic analysis of beams subjected to arbitrary (distributed or concentrated) transverse moving, as well as to axial loading.
(ii) The homogeneous linear half-space is approximated by a tensionless three-parameter viscoelastic foundation.
(iii) The beam is supported by the most general nonlinear boundary conditions including elastic support or restrain.
(iv) The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as shear forces along the span induced by the applied axial loading.
(v) The shear deformation coefficients are evaluated using an energy approach [28]. This approach leads to more realistic values for these coefficients, since the Timoshenko's [29] and Cowper's [30] definitions often lead to unsatisfactory results [31,32], while definitions given by other researchers [33,34] lead to negative values for these factors.
(vi) The effect of the material's Poisson ratio v is taken into account in the calculation of the shear deformation coefficients [28,35].
(vii) The proposed method employs a BEM approach (requiring boundary discretization) resulting in line or parabolic elements instead of area elements of the Finite Element solutions (requiring the whole cross-section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Analyses are performed to illustrate, wherever possible, the accuracy of the developed method, to investigate the effects of various parameters, such as the load velocity, load frequency, shear deformation, foundation nonlinearity, damping, on the beam displacements and stress resultants and to examine how the consideration of shear and axial compression affects the response of the system.

2. Statement of the problem

Let us consider a prismatic beam of length l (Fig. 1a), of arbitrary constant cross-section of area A (Fig. 1b), having at least one axis of symmetry (z-axis). The homogeneous isotropic and linearly elastic material of the beam cross-section, with modulus of elasticity E, shear modulus G and Poisson's ratio v occupies the two-dimensional multiply connected region Ω of the y, z plane and is bounded by the $\Gamma_j (j=1, 2, \ldots, K)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. The beam is supported on a homogeneous tensionless nonlinear three-parameter viscoelastic soil. The foundation model is characterized by the linear Winkler modulus k_L, the nonlinear Winkler modulus k_{NL}, the Pasternak (shear) foundation modulus k_P and the damping coefficient c. Taking into account the unbonded contact between beam and soil, the interaction pressure at the interface can be only compressive and is represented for the transverse direction by the following relations:

$$p_{sz}(x,t) = \tilde{H}(x,t) p_{\mathrm{react}}(x,t) \tag{1a}$$

$$p_{\mathrm{react}}(x,t) = k_L w(x,t) + k_{\mathrm{NL}} w^3(x,t) - k_P \frac{\partial^2 w(x,t)}{\partial x^2} + c \frac{\partial w(x,t)}{\partial t} \tag{1b}$$

where $\tilde{H}(x,t)$ is a unit step function defined as

$$\tilde{H}(x,t) = \begin{cases} 1 & \text{if } p_{\mathrm{react}}(x,t) > 0 \\ 0 & \text{if } p_{\mathrm{react}}(x,t) \leq 0 \end{cases} \tag{2}$$

The foundation reaction p_{react} of Eq. (1b) takes into account the nonlinear behavior of the soil (e.g. ballast and rail-bed) as proposed by Dahlberg [3], demonstrating that the differences between the nonlinear and linear models are considerable

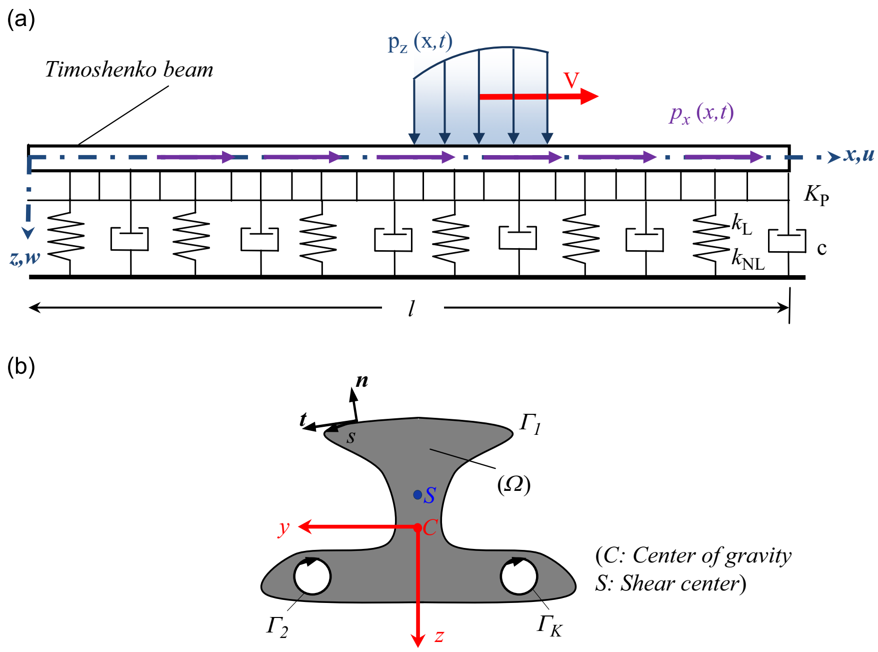

Fig. 1. Prismatic beam resting on a nonlinear viscoelastic foundation (a) of an arbitrary monosymmetric cross-section occupying the two-dimensional region Ω (b).

and a nonlinear track model simulates the rail deflection quite well whereas the equivalent linear one cannot. Later, Wu and Thompson [36] presented a similar nonlinear model and studied the problem of wheel/track impact employing the finite element method. Moreover, for real sample of the hardening behavior of the foundation one can refer to [37] where detailed field measurement results are presented.

The beam is subjected to the combined action of the arbitrarily distributed or concentrated transverse along the axis of symmetry moving loading $p_z=p_z(x,t)$ with constant velocity V as well as to axial loading $p_x=p_x(x,t)$, as shown in Fig. 1a. Under the action of this loading, the displacement field of the beam taking into account shear deformation effect is given as

$$\bar{u}(x,t,t)=u(x,t)+z\theta_y(x,t) \tag{3a}$$

$$\bar{w}(x,t)=w(x,t) \tag{3b}$$

where $\bar{u}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes (Fig. 1b); $u(x,t)$, $w(x,t)$ are the corresponding components of the centroid C and $\theta_y(x,t)$ is the angle of rotation due to bending of the cross-section with respect to the same point.

Employing the strain–displacement relations of the three-dimensional elasticity for moderate displacements the following strain components can be easily obtained:

$$\varepsilon_{xx}=\frac{\partial\bar{u}}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial\bar{u}}{\partial x}\right)^2+\left(\frac{\partial\bar{w}}{\partial x}\right)^2\right]\approx\frac{\partial\bar{u}}{\partial x}+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial x}\right)^2 \tag{4a}$$

$$\gamma_{xz}=\frac{\partial\bar{w}}{\partial x}+\frac{\partial\bar{u}}{\partial z}+\left(\frac{\partial\bar{u}}{\partial x}\frac{\partial\bar{u}}{\partial z}+\frac{\partial\bar{w}}{\partial x}\frac{\partial\bar{w}}{\partial z}\right)\approx\frac{\partial\bar{w}}{\partial x}+\frac{\partial\bar{u}}{\partial z}+\frac{\partial\bar{w}}{\partial x}\frac{\partial\bar{w}}{\partial z} \tag{4b}$$

$$\varepsilon_{zz}=\gamma_{yz}=0 \tag{4c}$$

since for moderate displacements it is $(\partial\bar{u}/\partial x)^2 \ll \partial\bar{u}/\partial x$ and $(\partial\bar{u}/\partial x)(\partial\bar{u}/\partial z) \ll (\partial\bar{w}/\partial x)+(\partial\bar{u}/\partial z)$. Substituting the displacement components (3) to the strain–displacement relations (4), the strain components can be written as

$$\varepsilon_{xx}=u'+z\theta_y'+\tfrac{1}{2}w'^2 \tag{5a}$$

$$\gamma_{xz}=w'+\theta_y \tag{5b}$$

where $(')$ denotes differentiation with respect to x and γ_{xz} is the additional angle of rotation of the cross-section due to shear deformation.

Considering strains to be small, employing the second Piola–Kirchhoff stress tensor and assuming an isotropic and homogeneous material, the stress components are defined in terms of displacement components by employing Eqs. (5) as

$$S_{xx}=E(u'+z\theta_y'+\tfrac{1}{2}w'^2) \tag{6a}$$

$$S_{xz} = G(w' + \theta_y) \tag{6b}$$

The governing equations and the boundary conditions of the beam subjected to nonlinear vibrations are obtained employing Hamilton's principle, defined as

$$\delta \int_{t_1}^{t_2} (U_{\text{int}} + U_b - K - W_{\text{ext}})\,dt = 0 \tag{7}$$

and expressed as a function of the stress resultants acting on the cross-section of the beam in the deformed state. In Eq. (7), $\delta(\cdot)$ denotes variation of quantities, while U_{int} is the strain energy of the beam due to normal and shear stresses, U_b is the strain energy of the elastic boundary conditions and K and W_{ext} are the kinetic energy and the external load work, respectively, given as

$$\delta U_{\text{int}} = \int_V (S_{xx}\delta\varepsilon_{xx} + S_{xz}\delta\gamma_{xz})\,dV \tag{8a}$$

$$\delta U_b = \delta\left(\frac{1}{2}\sum_b^2 (k_u^b u_b^2 + k_w^b w_b^2 + k_\theta^b \theta_{yb}^2)\right) \tag{8b}$$

$$\delta K = \frac{1}{2}\int_V \rho(\delta\dot{\bar{u}}^2 + \delta\dot{\bar{w}}^2)\,dV \tag{8c}$$

$$\delta W_{\text{ext}} = \int_L (p_x\delta u + p_z\delta w - \delta(W_{p_{sz}}))\,dx + \sum_b^2 (N_x^b\delta u_b + V_z^b\delta w_b + M_y^b\delta\theta_{yb}) \tag{8d}$$

where $(\dot{\ })$ denotes differentiation with respect to time t, while k_u^b, k_w^b, k_θ^b are the translational and rotational springs and N_x^b, V_z^b, M_y^b are the externally applied forces and moments at the boundaries. Substituting the expressions of the stress components (6) into the well-known stress resultants equations, the latter are obtained as

$$N = EA\left(u' + \tfrac{1}{2}w'^2\right) \tag{9a}$$

$$M_y = EI_y\theta_y' \tag{9b}$$

$$V_z = GA_z(w' + \theta_y) \tag{9c}$$

where A is the cross-section area, I_y the moment of inertia with respect to the z-axis and GA_z is its shear rigidity of the Timoshenko's beam theory, where

$$A_z = \kappa_z A = \frac{1}{a_z}A \tag{10}$$

is the shear area, respectively, with κ_z the shear correction factor and a_z the shear deformation coefficient. Substituting the stress components given in Eqs. (6) and the strain ones given in Eqs. (5) to the strain energy variation δU_{int} (Eqs. (8a)) and employing Eq. (7), the equilibrium equations of the beam are derived as

$$\rho A\ddot{u} - EA(u'' + w'w'') = p_x \tag{11a}$$

$$\rho A\ddot{w} - (Nw')' - GA_z(w'' + \theta_y') + p_{sz} = p_z \tag{11b}$$

$$EI_y\theta_y'' - GA_z(w' + \theta_y) = \rho I_y\ddot{\theta}_y \tag{11c}$$

Combining Eqs. (11b) and (11c) the following differential equations with respect to u, w are derived:

$$\rho A\ddot{u} - EA(u'' + w'w'') = p_x \tag{12a}$$

$$EI_y w'''' + \rho A\ddot{w} + p_{sz} + \frac{EI_y}{GA_z}((Nw')''' - \rho A(\ddot{w})'' - p_{sz}'' + p_z'') - (Nw')' - \rho I_y(\ddot{w})'' - \frac{\rho I_y}{GA_z}([(Nw')']^{\cdot\cdot} - \rho A\ddddot{w} - \ddot{p}_{sz} + \ddot{p}_z) = p_z \tag{12b}$$

Eqs. (12a) and (12b) constitute the governing differential equations of a Timoshenko–Rayleigh beam, supported on a tensionless nonlinear three-parameter viscoelastic foundation, subjected to nonlinear vibrations due to the combined action of arbitrarily distributed or concentrated transverse moving loading as well as to axial loading. These equations are also subjected to the pertinent boundary conditions of the problem at hand given as

$$a_1 u(x,t) + \alpha_2 N(x,t) = \alpha_3 \tag{13}$$

$$\beta_1 w(x,t) + \beta_2 V_z(x,t) = \beta_3, \quad \gamma_1\theta_y(x,t) + \gamma_2 M_y(x,t) = \gamma_3 \tag{14a,b}$$

at the beam ends $x=0,l$, together with the initial conditions

$$u(x,0)=\overline{u}_0(x), \quad \dot{u}(x,0)=\dot{\overline{u}}_0(x) \tag{15a,b}$$

$$w(x,0)=\overline{w}_0(x), \quad \dot{w}(x,0)=\dot{\overline{w}}_0(x) \tag{16a,b}$$

where $\overline{u}_0(x)$, $\overline{w}_0(x)$, $\dot{\overline{u}}_0(x)$ and $\dot{\overline{w}}_0(x)$ are the prescribed functions. In Eqs. (14) V_z and M_y are the reaction and bending moments, which together with the angle of rotation due to bending θ_y are given as

$$V_z = Nw' - EI_y w''' - \frac{EI_y}{GA_z}\left[(Nw')'' + p_z' - p_{sz}' - \rho A \frac{\partial \ddot{w}}{\partial x}\right] - \rho I_y \ddot{\theta}_y \tag{17a}$$

$$M_y = -EI_y w'' - \frac{EI_y}{GA_z}\left[(Nw')' + p_z - p_{sz} - \rho A \ddot{w}\right] \tag{17b}$$

$$\theta_y = \frac{EI_y}{G^2 A_z^2}\left(-p_z' + p_{sz}' - (Nw')'' + \rho A \frac{\partial \ddot{w}}{\partial x}\right) - \frac{1}{GA_z}(EI_y w''' + \rho I_y \ddot{\theta}_y + GA_z w') \tag{17c}$$

Finally, α_k, β_k, γ_k $(k=1,2,3)$ are functions specified at the beam ends $x=0,l$. Eqs. (13)–(14) describe the most general nonlinear boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1=\beta_1=1$, $\gamma_1=1$, $\alpha_2=\alpha_3=\beta_2=\beta_3=\gamma_2=\gamma_3=0$).

The solution of the initial boundary value problem given from Eqs. (12a) and (12b), subjected to the boundary conditions (13)–(14) and the initial conditions (15)–(16), which represents the geometrically nonlinear (large deflection) flexural dynamic analysis of a Timoshenko–Rayleigh beam, supported on a tensionless nonlinear three-parameter viscoelastic foundation, presumes the evaluation of the shear deformation coefficient a_z corresponding to the principal centroidal system of axes Cyz. This coefficient is established equating the approximate formula of the shear strain energy per unit length [33]

$$U_{\text{appr.}} = \frac{a_z Q_z^2}{2AG} \tag{18}$$

with the exact one given from

$$U_{\text{exact}} = \int_\Omega \frac{(\tau_{xz})^2}{2G}\, \mathrm{d}\Omega \tag{19}$$

and are obtained as [28]

$$a_z = \frac{1}{\kappa_z} = \frac{A}{\varDelta^2}\int_\Omega [(\boldsymbol{\nabla}\Phi) - \mathbf{d}]\cdot[(\boldsymbol{\nabla}\Phi) - \mathbf{d}]\,\mathrm{d}\Omega \tag{20}$$

where (τ_{xz}) is the transverse (direct) shear stress component, $(\boldsymbol{\nabla}) \equiv \mathbf{i}_y(\partial/\partial y) + \mathbf{i}_z(\partial/\partial z)$ is a symbolic vector with $\mathbf{i}_y$, $\mathbf{i}_z$ the unit vectors along the y and z axes, respectively, $\varDelta$ is given from

$$\varDelta = 2(1+v)I_y I_z \tag{21}$$

v is the Poisson ratio of the cross-section material and $\mathbf{d}$ is a vector defined as

$$\mathbf{d} = (vI_z yz)\mathbf{i}_y - \left(vI_z \frac{y^2 - z^2}{2}\right)\mathbf{i}_z \tag{22}$$

and $\Phi(y,z)$ is a stress function evaluated from the solution of the following Neumann type boundary value problem [28]:

$$\nabla^2 \Phi = -2I_z z \quad \text{in } \Omega \tag{23a}$$

$$\frac{\partial \Phi}{\partial n} = \mathbf{n}\cdot\mathbf{d} \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \tag{23b}$$

where $\mathbf{n}$ is the outward normal vector to the boundary Γ. In the case of negligible shear deformations $a_z=0$. It is also worth noting here that the boundary conditions (23b) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\mathbf{n}$ vanishes on the free surface of the beam.

3. Integral representations—numerical solution

3.1. Axial displacement u and transverse displacement w

According to the precedent analysis, the nonlinear flexural dynamic analysis of Timoshenko–Rayleigh beams, supported on a tensionless nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections reduces in establishing the displacement components $u(x,t)$ and $w(x,t)$ having continuous derivatives, respectively, up to the second and up to the fourth order with respect to x and also having derivatives up to the second order with respect to t (ignoring the inertia terms of the fourth order [38]). These displacement components must satisfy the coupled governing differential Eqs. (12a) and (12b) inside the beam, the boundary conditions (13)–(14) at the beam ends $x=0,l$ and the initial conditions (15)–(16).

Eqs. (12a) and (12b) are solved using the Analog Equation Method [27] as this is developed for hyperbolic differential equations [39]. According to this method, let $u(x,t)$, $w(x,t)$ be the sought solution of the aforementioned boundary value problem. Setting $u_1=u(x,t)$ and $u_2(x,t)=w(x,t)$ and differentiating with respect to x these functions two and four times, respectively, yields

$$\frac{\partial^2 u_1}{\partial x^2}=q_1(x,t), \quad \frac{\partial^4 u_2}{\partial x^4}=q_2(x,t) \tag{24a,b}$$

Eqs. (24a) and (24b) are quasi-static, that is the time variable appears as a parameter. They indicate that the solution of Eqs. (12a) and (12b) can be established by solving Eqs. (24a) and (24b) under the same boundary conditions (13)–(14), provided that the fictitious load distributions $q_i(x,t)$ $(i=1,2)$ are first established. These distributions can be determined using BEM as follows.

The solutions of Eqs. (24a) and (24b) are given in integral form as [39]

$$u_1(x,t)=\int_0^l q_1 u_1^* \,\mathrm{d}x-\left[u_1^*\frac{\partial u_1}{\partial x}-\frac{\mathrm{d}u_1^*}{\mathrm{d}x}u_1\right]_0^l \tag{25a}$$

$$u_2(x,t)=\int_0^l q_2 u_2^* \,\mathrm{d}x-\left[u_2^*\frac{\partial^3 u_2}{\partial x^3}-\frac{\mathrm{d}u_2^*}{\mathrm{d}x}\frac{\partial^2 u_2}{\partial x^2}+\frac{\mathrm{d}^2u_2^*}{\mathrm{d}x^2}\frac{\partial u_2}{\partial x}-\frac{\mathrm{d}^3u_2^*}{\mathrm{d}x^3}u_2\right]_0^l \tag{25b}$$

where u_1^* and u_2^* are the fundamental solutions given as

$$u_1^*=\frac{1}{2}|r|, \quad u_2^*=\frac{1}{12}l^3\left(2+\left|\frac{r}{l}\right|^3-3\left|\frac{r}{l}\right|^2\right) \tag{26a,b}$$

with $r=x-\xi$, x,ξ points of the beam, which are particular singular solutions of the equations

$$\frac{\mathrm{d}^2u_1^*}{\mathrm{d}x^2}=\delta(x-\xi), \quad \frac{\mathrm{d}^4u_2^*}{\mathrm{d}x^4}=\delta(x-\xi) \tag{27a,b}$$

Employing Eqs. (26a) and (26b), the integral representations (25a) and (25b) can be written as

$$u_1(x,t)=\int_0^l q_1\left(\Lambda_2(r)+\frac{1}{2}l\right)\mathrm{d}x-\left[\left(\Lambda_2(r)+\frac{1}{2}l\right)\frac{\partial u_1}{\partial x}+\Lambda_1(r)u_1\right]_0^l \tag{28a}$$

$$u_2(x,t)=\int_0^l q_2\Lambda_4(r)\,\mathrm{d}x-\left[\Lambda_4(r)\frac{\partial^3 u_2}{\partial x^3}+\Lambda_3(r)\frac{\partial^2 u_2}{\partial x^2}+\Lambda_2(r)\frac{\partial u_2}{\partial x}+\Lambda_1(r)u_2\right]_0^l \tag{28b}$$

where $\Lambda_j(r)$ $(j=1,2,3,4)$ are the kernels. Note that in Eqs. (28a) and (28b), for the line integrals it is $r=x-\xi$, x and ξ points inside the beam, whereas for the rest terms it is $r=x-\zeta$, x inside the beam and ζ at the beam ends $0,l$.

Differentiating Eqs. (28a) and (28b) with respect to x results in the integral representations of the derivatives of u_i as

$$\frac{\partial u_1(x,t)}{\partial x}=\int_0^l q_1\Lambda_1(r)\,\mathrm{d}x-\left[\Lambda_1(r)\frac{\partial u_1}{\partial x}\right]_0^l \tag{29a}$$

$$\frac{\partial^2 u_1(x,t)}{\partial x^2}=q_1(x,t) \tag{29b}$$

$$\frac{\partial u_2(x,t)}{\partial x}=\int_0^l q_2\Lambda_3(r)\,\mathrm{d}x-\left[\Lambda_3(r)\frac{\partial^3 u_2}{\partial x^3}+\Lambda_2(r)\frac{\partial^2 u_2}{\partial x^2}+\Lambda_1(r)\frac{\partial u_2}{\partial x}\right]_0^l \tag{29c}$$

$$\frac{\partial^2 u_2(x,t)}{\partial x^2}=\int_0^l q_2\Lambda_2(r)\,\mathrm{d}x-\left[\Lambda_2(r)\frac{\partial^3 u_2}{\partial x^3}+\Lambda_1(r)\frac{\partial^2 u_2}{\partial x^2}\right]_0^l \tag{29d}$$

$$\frac{\partial^3 u_2(x,t)}{\partial x^3}=\int_0^l q_2\Lambda_1(r)\,\mathrm{d}x-\left[\Lambda_1(r)\frac{\partial^3 u_2}{\partial x^3}\right]_0^l \tag{29e}$$

$$\frac{\partial^4 u_2(x,t)}{\partial x^4}=q_2(x,t) \tag{29f}$$

Integral representations (28a) and (28b) and (29a) and (29c) when applied to the beam ends (0,l), together with the boundary conditions (13)–(14), are employed to express the unknown boundary quantities $u_i(\zeta,t)$, $u_{i,x}(\zeta,t)$, $u_{2,xx}(\zeta,t)$ and $u_{2,xxx}(\zeta,t)$ ($\zeta=0,l$) in terms of q_i. This is accomplished numerically. More specifically, the interval (0,l) is divided into L elements, on which $q_i(x,t)$ is assumed to vary according to certain law (constant, linear, parabolic, etc.). The constant element assumption, which is employed here as the numerical implementation, becomes very simple and the obtained results are very good.

Employing the aforementioned procedure, the following set of 12 nonlinear algebraic equations is obtained:

$$\begin{bmatrix} \mathbf{T}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{D}_1^{\mathrm{nl}} \\ \mathbf{D}_2^{\mathrm{nl}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{a}_3 \\ \mathbf{b}_3 \end{Bmatrix} \tag{30}$$

with

$$\mathbf{T}_{11} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{0} & \mathbf{D}_{22} & \mathbf{D}_{23} \end{bmatrix}, \quad \mathbf{T}_{22} = \begin{bmatrix} \mathbf{F}_3 & \mathbf{E}_{35} & \mathbf{E}_{36} & \mathbf{E}_{37} & \mathbf{E}_{38} \\ \mathbf{F}_4 & \mathbf{0} & \mathbf{E}_{46} & \mathbf{E}_{47} & \mathbf{E}_{48} \\ \mathbf{0} & \mathbf{D}_{55} & \mathbf{D}_{56} & \mathbf{D}_{57} & \mathbf{D}_{58} \\ \mathbf{0} & \mathbf{D}_{65} & \mathbf{D}_{66} & \mathbf{0} & \mathbf{D}_{68} \end{bmatrix} \tag{31a,b}$$

where $\mathbf{D}_{22}$–$\mathbf{D}_{68}$ are 2×2 rectangular known, in general time dependent, matrices including the values of the functions a_j, β_j, γ_j ($j=1,2$) of Eqs. (13)–(14); $\mathbf{D}_1^{\mathrm{nl}}$, $\mathbf{a}_3$ and $\mathbf{D}_2^{\mathrm{nl}}$, $\mathbf{b}_3$ are 4×1 and 8×1, respectively, known, in general time dependent, column matrices including the boundary values of the functions a_3, β_3, γ_3 of Eqs. (13)–(14); $\mathbf{E}_{12}$, $\mathbf{E}_{13}$, $\mathbf{E}_{35}$, $\mathbf{E}_{36}$, $\mathbf{E}_{37}$, $\mathbf{E}_{38}$, $\mathbf{E}_{46}$, $\mathbf{E}_{47}$, $\mathbf{E}_{48}$ are rectangular 2×2 known coefficient matrices resulting from the values of the kernels $\Lambda_j(r)$ ($j=1,2,3,4$) at the beam ends and $\mathbf{F}_1$, $\mathbf{F}_3$, $\mathbf{F}_4$ are $2\times L$ rectangular known matrices originating from the integration of the kernels on the axis of the beam, while the generalized unknown vectors $\mathbf{d}_1^{\mathrm{T}} = \{\, \mathbf{q}_1 \quad \hat{\mathbf{u}}_1 \quad \hat{\mathbf{u}}_{1,x} \,\}$ and $\mathbf{d}_2^{\mathrm{T}} = \{\, \mathbf{q}_2 \quad \hat{\mathbf{u}}_2 \quad \hat{\mathbf{u}}_{2,x} \quad \hat{\mathbf{u}}_{2,xx} \quad \hat{\mathbf{u}}_{2,xxx} \,\}$ are vectors including the L unknown time dependent nodal values of the fictitious loads $q_i = \{q_1^i\ q_2^i \ldots q_L^i\}^{\mathrm{T}}$ ($i=1,2$) and the vectors including the two unknown time dependent boundary values of the respective boundary quantities

$$\hat{\mathbf{u}}_i = \{u_i(0,t) \quad u_i(l,t)\}^{\mathrm{T}}, \quad (i=1,\,2) \tag{32a}$$

$$\hat{\mathbf{u}}_{i,x} = \left\{ \frac{\partial u_i(0,t)}{\partial x} \quad \frac{\partial u_i(l,t)}{\partial x} \right\}^{\mathrm{T}}, \quad (i=1,\,2) \tag{32b}$$

$$\hat{\mathbf{u}}_{2,xx} = \left\{ \frac{\partial^2 u_2(0,t)}{\partial x^2} \quad \frac{\partial^2 u_2(l,t)}{\partial x^2} \right\}^{\mathrm{T}} \tag{32c}$$

$$\hat{\mathbf{u}}_{2,xxx} = \left\{ \frac{\partial^3 u_2(0,t)}{\partial x^3} \quad \frac{\partial^3 u_2(l,t)}{\partial x^3} \right\}^{\mathrm{T}} \tag{32d}$$

Discretization of Eqs. (28) and (29) and application to the L collocation nodal points yields

$$\mathbf{u}_1 = \mathbf{A}_1^0\mathbf{q}_1 + \mathbf{C}_0\hat{\mathbf{u}}_1 + \mathbf{C}_1\hat{\mathbf{u}}_{1,x}, \quad \mathbf{u}_{1,x} = \mathbf{A}_1^1\mathbf{q}_1 + \mathbf{C}_0\hat{\mathbf{u}}_{1,x}, \quad \mathbf{u}_{1,xx} = \mathbf{q}_1 \tag{33a,b,c}$$

$$\mathbf{u}_2 = \mathbf{A}_2^0\mathbf{q}_2 + \mathbf{C}_0\hat{\mathbf{u}}_2 + \mathbf{C}_1'\hat{\mathbf{u}}_{2,x} + \mathbf{C}_2\hat{\mathbf{u}}_{2,xx} + \mathbf{C}_3\hat{\mathbf{u}}_{2,xxx} \tag{33d}$$

$$\mathbf{u}_{2,x} = \mathbf{A}_2^1\mathbf{q}_2 + \mathbf{C}_0\hat{\mathbf{u}}_{2,x} + \mathbf{C}_1'\hat{\mathbf{u}}_{2,xx} + \mathbf{C}_2\hat{\mathbf{u}}_{2,xxx} \tag{33e}$$

$$\mathbf{u}_{2,xx} = \mathbf{A}_2^2\mathbf{q}_2 + \mathbf{C}_0\hat{\mathbf{u}}_{2,xx} + \mathbf{C}_1'\hat{\mathbf{u}}_{2,xxx} \tag{33f}$$

$$\mathbf{u}_{2,xxx} = \mathbf{A}_2^3\mathbf{q}_2 + \mathbf{C}_0\hat{\mathbf{u}}_{2,xxx}, \quad \mathbf{u}_{2,xxxx} = \mathbf{q}_2 \tag{33g,h}$$

where $\mathbf{A}_1^i$, $\mathbf{A}_2^j$ ($i=0,1$), ($j=0,1,2,3$) are $L\times L$ known matrices; $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_1'$, $\mathbf{C}_2$, $\mathbf{C}_3$ are $L\times2$ known matrices and $\mathbf{u}_i$, $\mathbf{u}_{i,x}$, $\mathbf{u}_{i,xx}$, $\mathbf{u}_{i,xxx}$, $\mathbf{u}_{i,xxxx}$ are time dependent vectors including the values of $u_i(x,t)$ and their derivatives at the L nodal points. Eqs. (33a) (33f) and (33g) can be assembled more conveniently as

$$\mathbf{u}_1 = \mathbf{H}_1^0\mathbf{d}_1, \quad \mathbf{u}_{1,x} = \mathbf{H}_1^1\mathbf{d}_1 \tag{34a,b}$$

$$\mathbf{u}_2 = \mathbf{H}_2^0\mathbf{d}_2, \quad \mathbf{u}_{2,x} = \mathbf{H}_2^1\mathbf{d}_2, \quad \mathbf{u}_{2,xx} = \mathbf{H}_2^2\mathbf{d}_2, \quad \mathbf{u}_{2,xxx} = \mathbf{H}_2^3\mathbf{d}_2 \tag{34c,d,e,f}$$

where $\mathbf{H}_1^i$,$\mathbf{H}_2^j$ ($i=0,1$), ($j=0,1,2,3$) are, respectively, $L\times(L+4)$ and $L\times(L+8)$ known matrices arising from $\mathbf{A}_1^i$, $\mathbf{A}_2^j$, $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_1'$, $\mathbf{C}_2$, $\mathbf{C}_3$.

Applying Eqs. (12a) and (12b) to the L collocation points and employing Eqs. (34), $2L$ semidiscretized nonlinear equations of motion are formulated as

$$\mathbf{M}\begin{Bmatrix}\ddot{\mathbf{d}}_1\\ \ddot{\mathbf{d}}_2\end{Bmatrix}+\mathbf{C}\begin{Bmatrix}\dot{\mathbf{d}}_1\\ \dot{\mathbf{d}}_2\end{Bmatrix}+\mathbf{K}\begin{Bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{Bmatrix}+\mathbf{k}^{\mathrm{nl}}(\mathbf{H}_1^i,\mathbf{H}_2^j,\mathbf{d}_1,\mathbf{d}_2)=\mathbf{f} \tag{35}$$

where $\mathbf{k}^{\mathrm{nl}}$ is a nonlinear generalized stiffness vector and $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{f}$ are generalized mass, damping, stiffness matrices and force vector, respectively.

Eqs. (34a) and (34c) when combined with Eqs. (15a)–(16a) and (15b)–(16b) yield the following $2L$ linear equations with respect to $\mathbf{d}_1$, $\mathbf{d}_2$ and $\dot{\mathbf{d}}_1$, $\dot{\mathbf{d}}_2$ for $t=0$:

$$\mathbf{H}_1^0\mathbf{d}_1(0)=\overline{\mathbf{u}}_0,\quad \mathbf{H}_2^0\mathbf{d}_2(0)=\overline{\mathbf{w}}_0 \tag{36a,b}$$

$$\mathbf{H}_1^0\dot{\mathbf{d}}_1(0)=\dot{\overline{\mathbf{u}}}_0,\quad \mathbf{H}_2^0\dot{\mathbf{d}}_2(0)=\dot{\overline{\mathbf{w}}}_0 \tag{37a,b}$$

Eqs. (36a) and (36b), together with Eq. (30) written for $t=0$, form a set of $2L+12$ nonlinear algebraic equations, which are solved to establish the initial conditions $\mathbf{d}_1(0)$, $\mathbf{d}_2(0)$, while similarly Eqs. (37a) and (37b), together with 12 equations resulting after differentiating Eq. (30) with respect to time and writing them for $t=0$, form a set of $2L+12$ linear algebraic equations from which the initial conditions $\dot{\mathbf{d}}_1(0)$, $\dot{\mathbf{d}}_2(0)$ are established.

The aforementioned initial conditions along with Eqs. (30) and (35) form an initial value problem of differential–algebraic equations (DAE), which can be solved using any efficient solver. In this study, the Petzold–Gear method is used [40] after introducing new variables to reduce the order of system [41] and after differentiating Eq. (30) with respect to time to obtain an equivalent system with a value of system index ind=1 [41].

3.2. *Stress function Φ(y,z)*

The evaluation of the stress functions $\Phi(y,z)$ is accomplished using BEM as this is presented in Sapountzakis and Mokos [28].

Moreover, since the geometrically nonlinear flexural dynamic analysis of Timoshenko–Rayleigh beams, supported on a tensionless nonlinear three-parameter viscoelastic foundation is solved by the BEM, the domain integrals for the evaluation of the area, the bending moments of inertia and the shear deformation coefficients (Eq. (20)) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments, the product of inertia and the cross-section area can be written as

$$I_y=\int_\Gamma (yz^2n_y)\,\mathrm{d}s,\quad I_z=\int_\Gamma (zy^2n_z)\,\mathrm{d}s \tag{38a,b}$$

$$A=\frac{1}{2}\int_\Gamma (yn_y+zn_z)\,\mathrm{d}s \tag{38c}$$

while the shear deformation coefficient a_z is obtained from the relation

$$a_z=\frac{A}{\Delta^2}\left((4v+2)I_zI_{\Phi z}+\frac{1}{4}v^2I_z^2I_{ed}-I_{\Phi d}\right) \tag{39}$$

where

$$I_{\Phi d}=\int_\Gamma \Phi(\mathbf{n}\cdot\mathbf{d})\,\mathrm{d}s \tag{40a}$$

$$I_{ed}=\int_\Gamma\left(y^4zn_z+z^4yn_y+\frac{2}{3}y^2z^3n_z\right)\mathrm{d}s \tag{40b}$$

$$I_{\Phi z}=\frac{1}{6}\int_\Gamma[-2I_zz^4yn_y+(3\Phi n_z-z(\mathbf{n}\cdot\mathbf{d}))z^2]\,\mathrm{d}s \tag{40c}$$

4. Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible the accuracy and the range of applications of the developed method. In all the examples treated, the results have been obtained using $L=41$ nodal points (longitudinal discretization), 400 boundary elements (cross-section discretization) and a time step of $\Delta t=1.0\,\mu$s.

Example 1. For comparison reasons, in the first example the linear dynamic analysis of a simply supported steel Timoshenko beam free of foundation support is examined. The material and geometric constants are given in Table 1, while the shear deformation coefficient is considered as $a_z = 1.2$ corresponding to rectangular cross-section. The beam is subjected to a concentrated moving load with constant velocity, $p_z(x,t) = P\delta(x - Vt)$, $P = 700$ N, $V = 12$ km/h (δ is the Dirac's delta function). In Table 2, the maximum deflections of the midpoint of the beam for various internal nodal points' discretization schemes are presented, illustrating that convergence is achieved for a small number of nodal points. In Fig. 2, the time history of the deflection $w(l/2,t)$ at the beam's midpoint is presented as compared with the one obtained from a modal superposition method [20], demonstrating the accuracy of the results of the proposed method.

Example 2. As a second example, a simply supported Timoshenko–Rayleigh beam subjected to a concentrated moving load with constant velocity, $p_z(x,t) = P\delta(x - Vt)$ (δ is the Dirac's delta function), having the material, geometric and loading constants given in Table 3 and resting on a Pasternak viscoelastic foundation is considered. In Fig. 3 the time history of the

Table 1
Geometric constants of the beam of Example 1.

l (m)	10
E (GPa)	207
I_y (m^4)	1.04×10^{-6}
A (m^2)	10^{-3}
ρ (kg/m^3)	7040

Table 2
Maximum values of the deflection $w_{max}(l/2,t)$ of the beam of Example 1 and divergence values for various discretization schemes.

Nodal points	11	17	21	27	31	37	41
Max deflection (cm)	6.9578	6.9636	7.0158	7.0167	7.0617	7.0967	7.0968
Divergence (%)	1.96	1.88	1.14	1.13	0.5	0.001	–

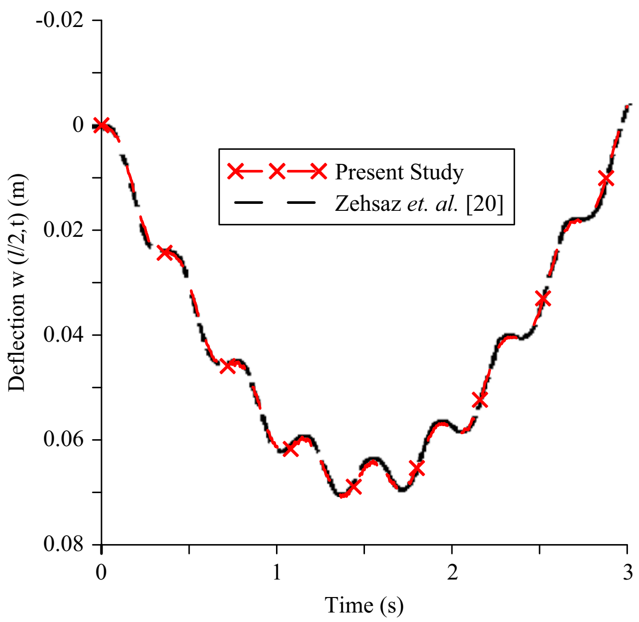

Fig. 2. Time history of the midpoint deflection $w(l/2,t)$ of the beam of Example 1.

Table 3
Geometric, foundation and loading constants of the beam of Example 2.

l (m)	10	a_z	1.176
E (GPa)	207	k_L (MPa)	20
I_y (m^4)	39.5×10^{-6}	k_P (kN)	69
A (m^2)	86.13×10^{-4}	c (kN s/m^2)	138
ρ (kg/m^3)	7820	P (kN)	144
v	0.3	V (km/h)	60

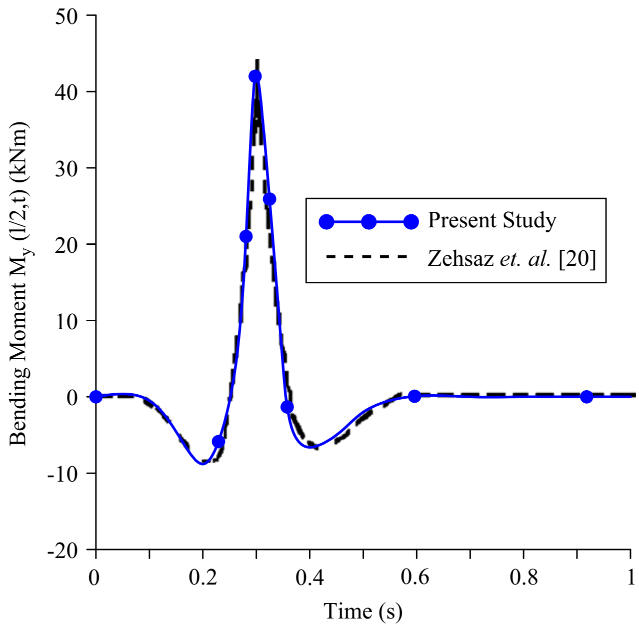

Fig. 3. Time history of the midpoint bending moment $M_y(l/2,t)$ of the beam of Example 2.

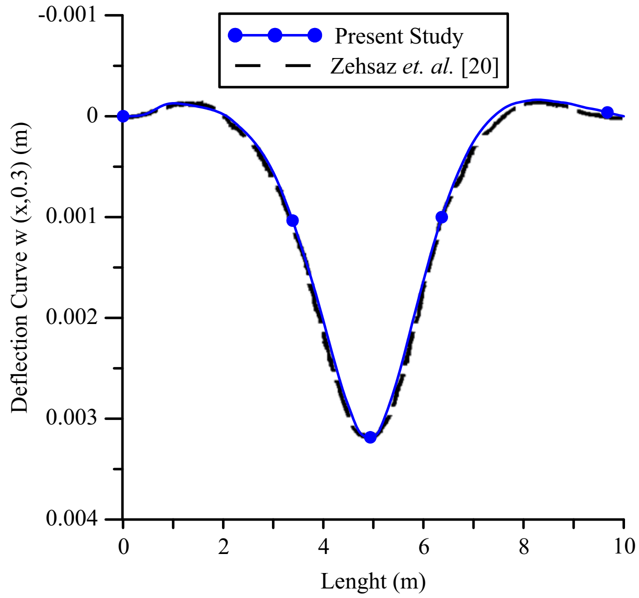

Fig. 4. Deflection curve $w(x,0.3)$, at the time instant $t=0.3$ s of the beam of Example 2.

bending moment $M_y(l/2,t)$ at the beam's midpoint and in Figs. 4 and 5 the deflection $w(x,0.3)$ and the bending moment $M_y(x,0.3)$ at the time instant $t=0.3s$ along the beam axis, respectively, are given. In these figures the corresponding curves obtained from the solution of the same beam resting on a Pasternak-type viscoelastic foundation (using a viscous shear layer) are also presented employing the modal superposition method considering the first ten modes [20]. From these figures, it is easily verified that due to the high value of the damping coefficient, the response of the beam approaches to zero after passage of the moving load. Moreover, the bed influence limiting the effect of the moving load to the nearby points is remarked. Finally, in Table 4 the absolute maximum deflection w_{max} as well as the maximum deflection $w_{max,l/2}$ and bending moment $M_{ymax,l/2}$ at the midpoint of the beam are presented for various values of the velocity for both Pasternak and Winkler foundations, taking into account shear deformation effect. From this table the negligible, in this example, decrease of the transverse displacements and the bending moments arising from the shear foundation layer are remarked.

Example 3. In order to illustrate the importance of the geometrically nonlinear (large deflection) analysis, a simply supported UIC60 rail track, resting on a three-parameter viscoelastic bilateral foundation is examined. The geometric, foundation and loading constants of the track are given in Table 5. The track is subjected to a concentrated moving harmonic load $p_z(x,t)=P\delta(x-Vt)\sin(\Omega t)$, where P, Ω are the amplitude and the frequency of the harmonic load,

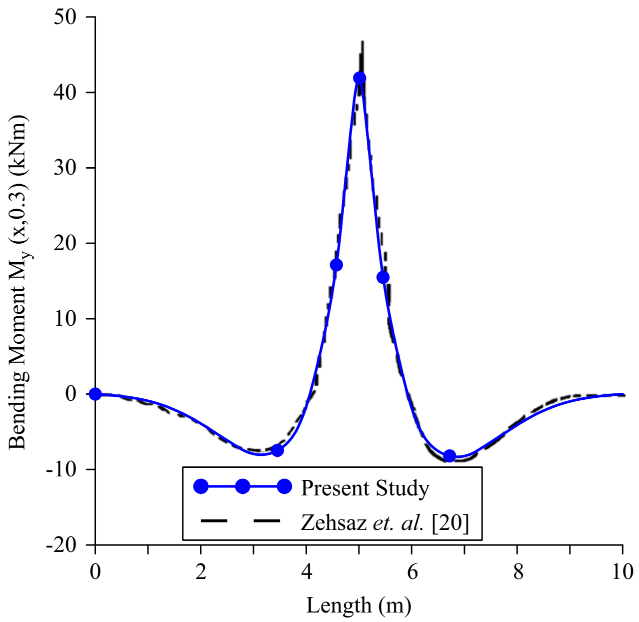

Fig. 5. Bending moment $M_y(x,0.3)$ along the beam axis, at the time instant $t=0.3$ s of the beam of Example 2.

Table 4
Absolute maximum deflection w_{max} (mm) and maximum values of the deflection $w_{max,l/2}$ (mm) and the bending moment $M_{ymax,l/2}$ (kN m) at the midpoint of the beam of Example 2, taking into account shear deformation effect.

V (km/h)	Pasternak			Winkler		
	w_{max}	$w_{max,l/2}$	$M_{ymax,l/2}$	w_{max}	$w_{max,l/2}$	$M_{ymax,l/2}$
0	3.318	3.171	38.37	3.327	3.185	38.43
10	3.335	3.189	39.91	3.346	3.200	40.92
60	3.345	3.202	41.17	3.350	3.209	41.24
100	3.466	3.281	41.89	3.473	3.285	42.40
120	3.522	3.363	45.99	3.531	3.369	46.17
150	3.649	3.465	48.01	3.657	3.471	48.19

Table 5
Geometric, foundation and loading constants of the UIC60 rail track [3,17] of Example 3.

l (m)	10	a_z	2.68
E (GPa)	210	k_L (MPa)	35
G (GPa)	77	k_{NL} (MN/m^4)	4×10^8
I_y (m^4)	30.55×10^{-6}	k_P (kN)	200
A (m^2)	76.86×10^{-4}	c (kN s/m^2)	145
ρ (kg/m^3)	7850	P (kN)	100

respectively, and δ is the Dirac's delta function. Moreover, the track is subjected to an either tensile or compressive distributed axial load $p_x(x,t)=\pm2500$(kN/m).

In Fig. 6 the time history and the extreme values of the central deflection $w(l/2,t)$ of the track resting on the viscoelastic Winkler foundation and subjected to a concentrated harmonic load at its midpoint ($V=0$ m/s, $\Omega=100$ rad/s) is presented, performing either a small or a large deflection analysis and taking into account both rotary inertia and shear deformation effect. To illustrate the significant effect of the load frequency, in Table 6 the maximum values of the deflection $w(l/2,t)$ at the midpoint of the track resting on the nonlinear three-parameter bilateral viscoelastic foundation, subjected to a harmonic moving load with constant velocity $V=100$ m/s are presented for various values of the excitation frequency Ω, performing either small or large deflection analysis (for both cases of tensile or compressive axial load). Moreover, in Table 7 the maximum deflections and bending moments of the track are presented for different types of bilateral viscoelastic foundation reaction, for $\Omega=400$ rad/s, $V=100$ m/s, while in Fig. 7 the deflection curves $w(x,0.055)$ along the track axis at the time instant $t=0.055$ s as well as their maximum values are also presented for both small and large deflection analyses for viscoelastic Winkler and three-parameter foundation. From the obtained results, it is concluded that the discrepancy between the geometrically linear and nonlinear analyses is not negligible and should not be ignored, while the influence of the shear deformation effect (increasing the transverse displacements and decreasing the bending

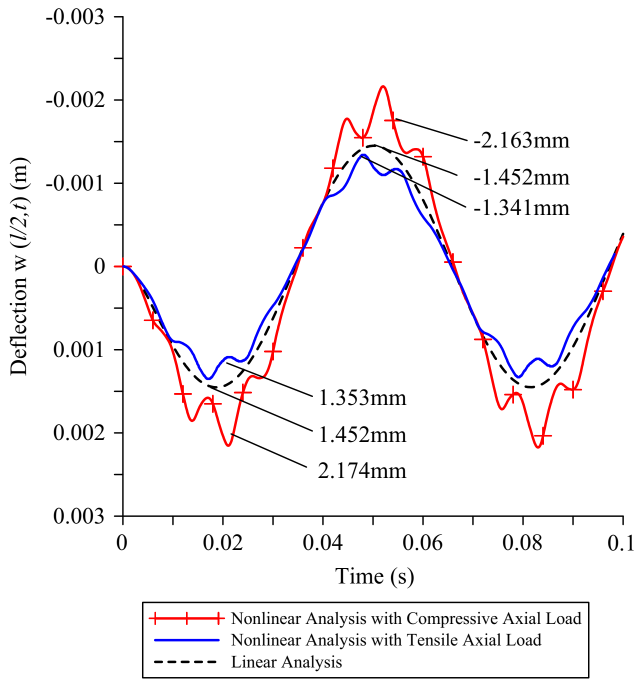

Fig. 6. Time history and extreme values of the midpoint deflection $w(l/2,t)$ of the track of Example 3.

Table 6
Maximum values of the deflection w (mm) at the midpoint of the track of Example 3, for various values of the excitation frequency Ω.

Ω (rad/s)	Linear	Nonlinear–tensile load	Nonlinear–compressive load
1.0	0.0715	0.0607	0.1009
5.0	0.2699	0.2338	0.3374
10	0.3999	0.3626	0.4693
50	0.4609	0.4445	0.5147
100	0.5690	0.5448	0.6272
200	0.5150	0.4724	0.5745
400	0.5782	0.5257	0.6589

Table 7
Maximum deflections $w_{\max}$ and bending moments $M_{y\max}$ of the track of Example 3, for different types of foundation reaction.

$w_{\max}$(mm)	Without shear deformation		With shear deformation	
$M_{y\max}$ (kN m)	Linear analysis	Nonlinear analysis	Linear analysis	Nonlinear analysis
Linear Winkler	0.9879 15.449	1.4336 22.869	0.9973 15.285	1.4436 22.805
Linear and nonlinear Winkler	0.5788 12.906	0.6859 15.917	0.5923 12.161	0.6992 15.916
Pasternak	0.9861 15.942	1.4235 22.779	0.9937 15.255	1.4452 22.735
Three-parameter	0.5783 12.887	0.6851 15.886	0.5820 12.148	0.6893 15.880

moments) in both of the aforementioned analyses is observed. This latter influence is more pronounced as the length of the track becomes smaller.

Example 4. To demonstrate the range of applications of the developed method, as a final application, a clamped-slide supported HEM240 beam of length $l=8$ m ($E=210$ GPa, $\nu=0.3$, $a_z=4.344$, $\rho=7.85$ tn/m^3) has been studied. The foundation model is characterized by the linear Winkler modulus $k_L=1.5$ MN/m^2, the nonlinear Winkler one $k_{NL}=1.5$ GN/m^4, the

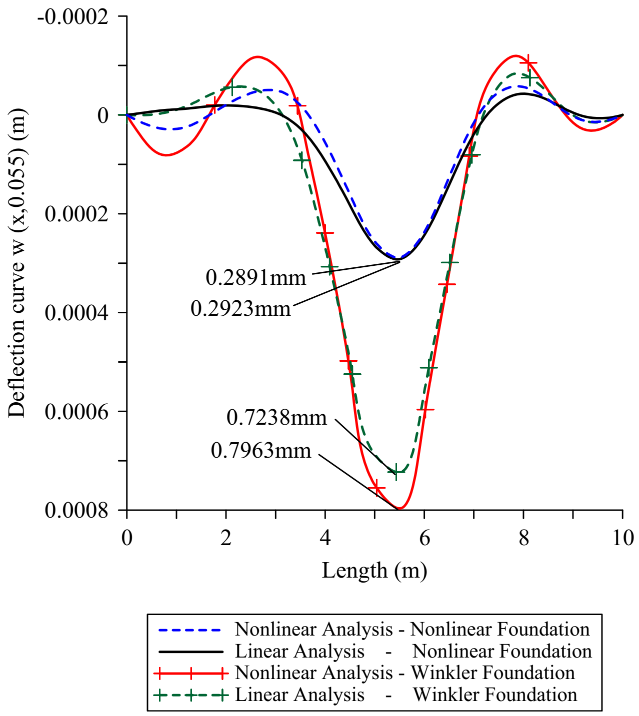

Fig. 7. Deflection curves $w(x,0.055)$, at the time instant $t=0.055$ s and their maximum values of the beam of Example 3.

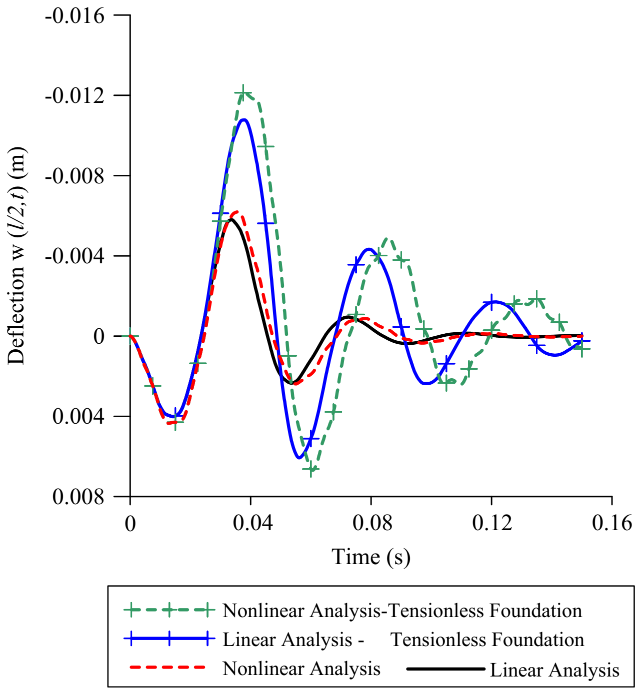

Fig. 8. Time history of the midpoint deflection $w(l/2,t)$ of the beam of Example 4 ($c=15$ kN s/m^2, $V=0$ m/s).

Pasternak (shear) modulus $k_P=400$ kN and the damping coefficient c (kN s/m^2). The beam is subjected to a uniformly distributed axial loading $p_x(x,t)=-1000$ kN/m and to a moving harmonic line load $p_z(x,t)$ of length $2a$ defined as

$$p_z(x,t)=P\frac{H(a^2-s^2)}{2a}\cos(\Omega t) \tag{41}$$

where P and Ω are the amplitude and the frequency of the harmonic load, H is the Heaviside step function and s is given as $s=(x+a)-Vt$. For this example, it is considered that $P=200$ kN, $\Omega=100$ rad/s and $a=0.381$ m.

In Fig. 8 the time history of the deflection $w(l/2,t)$ of the midpoint of the beam subjected to the aforementioned load ($c=15$ kN s/m^2, $V=0$ m/s) for the time period $0\leq t\leq 0.03$ s is presented, performing either a small or a large deflection

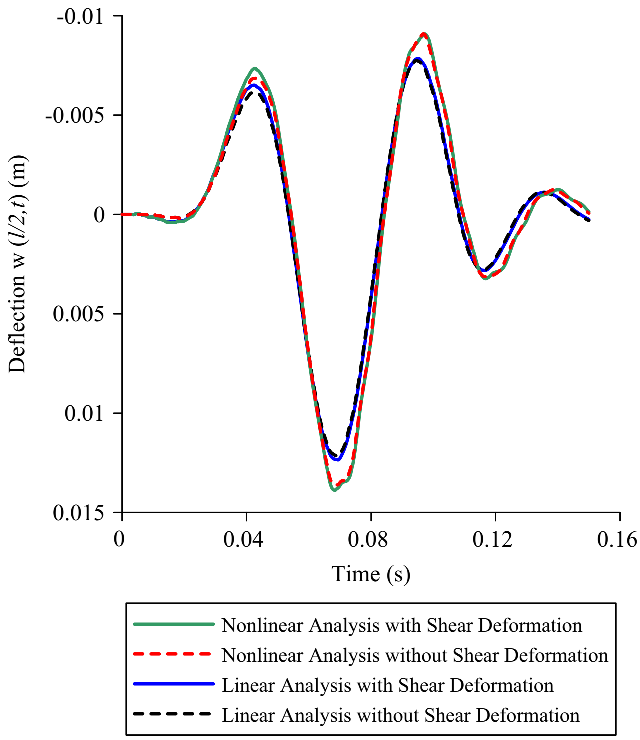

Fig. 9. Time history of the midpoint deflection $w(l/2,t)$ of the beam of Example 4 (c=15 kN s/m^2, V=80 m/s).

Table 8
Maximum values of the deflection w (cm) of the beam of Example 4.

	Without shear deformation		With shear deformation	
w_{max}	Linear analysis	Nonlinear analysis	Linear analysis	Nonlinear analysis
Nonlinear foundation				
c=0	1.985	2.373	2.019	2.448
c=5	1.678	1.966	1.715	2.023
c=10	1.471	1.681	1.487	1.713
c=15	1.323	1.472	1.335	1.484
c=20	1.197	1.305	1.210	1.314
c=40	0.8579	0.924	0.861	0.925
Nonlinear tensionless foundation				
c=0	2.473	3.094	2.568	3.376
c=5	2.169	2.544	2.254	2.651
c=10	1.925	2.251	2.001	2.352
c=15	1.726	2.014	1.794	2.104
c=20	1.561	1.819	1.619	1.896
c=40	1.128	1.298	1.165	1.343

analysis, taking into account rotary inertia, shear deformation effect and the tensionless character of the foundation, demonstrating the discrepancy between the conventional and the tensionless foundation. In Fig. 9 the time history of the deflection $w(l/2,t)$ of the midpoint of the beam subjected to the aforementioned load (c=15 kN s/m^2, V=80 m/s) is presented, performing either a small or a large deflection analysis and taking into account or ignoring both shear deformation effect and rotary inertia. Moreover, in Table 8 the maximum values of the deflection $w(x,t)$ of the beam resting on either a nonlinear or a tensionless nonlinear three-parameter foundation are also presented for various values of the damping coefficient c, performing both small and large deflection analyses and taking into account or ignoring shear deformation effect and rotary inertia. From the results obtained the influence of the damping coefficient is illustrated and the importance of large deflection analysis is verified.

5. Conclusions

In this paper, a boundary element method is developed for the geometrically nonlinear response of shear deformable beams of simply or multiply connected constant cross-section, traversed by moving loads, resting on tensionless nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections under general boundary conditions, taking into account the effects of shear deformation and rotary inertia. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse moving loading as well as to axial loading. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. The main conclusions that can be drawn from this investigation are

(a) The numerical technique presented in this investigation is well suited for computer aided analysis for beams of arbitrary simply or multiply connected cross-section having at least one axis of symmetry.
(b) The proposed method is developed for general dynamic moving loading, while the beam is subjected to the most general boundary conditions and rests on a linear or nonlinear viscoelastic foundation.
(c) The lift up of the beam caused by the tensionless character of the foundation is observed, leading to magnification of the consequences of the dynamic response.
(d) In some cases, the effect of shear deformation is significant, especially for low beam slenderness values, increasing the transverse displacements and decreasing the bending moments in both small and large deflection analyses.
(e) The discrepancy between the results of the linear and the nonlinear analyses is remarkable.
(f) The response of the beam is strongly influenced by the linear and nonlinear parameters of the foundation reaction.
(g) The damping coefficient is of paramount importance for beams on viscoelastic foundations, as it reduces the vibration amplitude and the consequences of the dynamic response.

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