Nonlinear Dynamic Analysis of Shear Deformable Beam-Columns on Nonlinear Three-Parameter Viscoelastic Foundation. I: Theory and Numerical Implementation

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Abstract: A boundary element method is developed for the nonlinear dynamic analysis of beam-columns of an arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on a nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections under general boundary conditions, taking into account the effects of shear deformation and rotary inertia. Part I is devoted to the theoretical development and numerical implementation of the method, while Part II discusses the examined numerical applications illustrating the efficiency (wherever possible), the accuracy, and the range of applications of the proposed method. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions, as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary-value problems are formulated with respect to the transverse displacements, axial displacement, and two stress functions, and solved using the analog equation method, a boundary element–based method. Application of the boundary element technique yields a nonlinear coupled system of equations of motion. The solution to this system is accomplished iteratively by employing the average acceleration method in combination with the modified Newton-Raphson method. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The proposed model takes into account the coupling effects of the bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. **DOI: 10.1061/(ASCE)EM.1943-7889.0000369.** *© 2013 American Society of Civil Engineers.*

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Introduction

The foundations of structures can be divided into shallow and deep foundations. Numerous research efforts have been made to improve the design and analysis of both of these cases and breakthroughs have been recognized in both the modeling of soil media as well as the simulation of the interaction between soil and the foundation. According to the modeling of the mechanical behavior of the subsoil and the soil-foundation interaction, assuming linear elastic, homogeneous, and isotropic soil behavior, the oldest, most famous and most frequently used mechanical model is the Winkler model. In this model the supporting soil behavior is approximated by a series of closely spaced, mutually independent, linear elastic vertical spring elements, providing resistance in direct proportion to the deflection of the beam. However, the application of this model is restricted to noncohesive soil media as a result of its inability to take into account the continuity or cohesion of the soil (i.e., the interaction between adjacent springs). To overcome this weakness, a second parameter is introduced (Filonenko-Borodich, Pasternak, or Hetenyi models) to account for the interactions among the linear elastic springs. The induction of this second parameter brings the modeling of the soil behavior closer to reality; however, its response is still not as complicated as the elastic continuum model. This fact resulted in the development of more sophisticated models comprising three independent parameters for the description of soil behavior. These three-parameter models constitute a generalization of the two-parameter models; the third parameter being used to make them more realistic and effective. More specifically, because in practice the support structure may be highly nonlinear (e.g., in many practical fields of railway engineering), the inclusion of a third parameter associated with the cubic nonlinearity of the deflection renders the arising mechanical model capable of distributing stresses correctly. Moreover, the three-parameter foundation, including material damping, is a very practical model for dynamic loading cases.

Keeping in mind that weight saving is of paramount importance in engineering structures, the study of the nonlinear effects on the analysis of supporting structural elements becomes essential. This nonlinearity results from retaining the square of the slope in strain-displacement relationships (intermediate nonlinear theory), avoiding

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in this way the inaccuracies arising from a linearized second-order analysis. Moreover, as a result of the intensive use of materials having a relatively high transverse shear modulus, the error incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading. Because analytic solutions for the aforementioned general beam problems are out of the question, the most commonly employed numerical methods are the finite-element method (FEM) (Lewandowski 1989), the boundary element method (BEM) (Sapountzakis and Kampitsis 2010), and the boundary integral equation method (BIEM) (Chen and Chen 2007).

When the beam-column deflections of a structure are small, a wide range of linear analysis tools, such as modal analysis, can be used, and some analytical results are possible. During the past few years, the linear dynamic analysis of beams on an elastic foundation has received a good amount of attention in the literature with the pioneer the work of Hetenyi (1966), who studied the elementary Bernoulli-Euler beams on an elastic Winkler foundation. Radeş (1972) presented the steady-state response of a finite rigid beam resting on a foundation defined by one inertial and three elastic parameters in the assumption of a permanent and smooth contact between the beam and foundation considering only uncoupled modes. Wang and Stephens (1977) studied the natural vibrations of a Timoshenko beam on a Pasternak-type foundation showing the effects of rotary inertia, shear deformation, and foundation constants of a beam employing general analytic solutions for simple cases of boundary conditions. Morgan and Sinha (1983) investigated the stability of Beck's column supported by three different viscoelastic foundations—namely, standard linear solid, Maxwell, and the Kelvin-Voigt—performing an exact dynamic analysis for each foundation model. Kuczma and Switka (1990) presented a solution algorithm for the analysis of unilateral, frictionless contact between a beam and a viscoelastic foundation. The problem was formulated in the form of a variational inequality, from which, after space discretization by the FEM, a linear complementary problem was derived. Subsequently, Wei and Yida (1994) analyzed the dynamic response of an elastic beam set on a linear viscoelastic Winkler foundation, impacted by a moving body at low velocity, while Matsunaga (1999), employing the method of power series expansion, presented the natural frequencies and buckling stresses of a deep beam-column on a two-parameter elastic foundation taking into account the effect of shear deformation, depth change, and rotary inertia. De Rosa (1995) and El-Mously (1999) derived explicit formulas for the fundamental natural frequencies of finite Timoshenko beams mounted on a finite Pasternak foundation. Sun (2001) employed Fourier transform to solve the problem of the steady-state response of a beam on a viscoelastic foundation subjected to a harmonic line load. Chen et al. (2001) established the dynamic stiffness matrix of an infinite or semiinfinite Timoshenko beam on a Winkler viscoelastic foundation subjected to a harmonic moving load. Coşkun (2003) studied the response of an elastic beam on a two-dimensional (2D) tensionless Pasternak foundation subjected to a central concentrated harmonic load using trigonometric-hyperbolic functions. Chen et al. (2004) proposed a mixed method combining the state-space method and the differential quadrature method for the bending and free vibration analysis of arbitrarily thick beams resting on a Pasternak elastic foundation. Kargarnovin and Younesian (2004) and Kargarnovin et al. (2005) presented the response of an infinite length Timoshenko beam of a uniform cross section, supported by either a generalized Pasternak-type or a nonlinear viscoelastic foundation and subjected to arbitrarily distributed harmonic moving loading and employing either complex Fourier transformation in conjunction with the residue and convolution integral theorems or a straightforward technique using the Lindstedt-Poincare perturbation method in conjunction with a Fourier integral transformation. Muscolino and Palmeri (2007) studied the response of beams resting on a viscoelastically damped foundation under moving single-degree-of-freedom (SDOF) oscillators through a novel state-space formulation, in which a number of internal variables are introduced with the aim of representing the frequency-dependent behavior of the viscoelastic foundation. Ying et al. (2008) derived an exact solution for the bending and free vibration analysis of functionally graded beams resting on a Winkler-Pasternak elastic foundation based on the 2D theory of elasticity and employing the state-space method. Çalim (2009) presented the dynamic behavior of Timoshenko beams on a Pasternak-type viscoelastic foundation subjected to time-dependent loading, employing the complementary functions method. Millán and Domínguez (2009) developed a simplified model for the analysis of the dynamic response of structures on piles and pile groups in viscoelastic or poroelastic soils under time harmonic excitation using a coupled BEM/FEM model able to take into account dynamic pile-soil-pile interactions in a rigorous manner. Finally, Younesian and Kargarnovin (2009) presented the dynamic response of infinite beams supported by a random viscoelastic Pasternak foundation subjected to harmonic moving loads, employing first-order perturbation theory and calculating appropriate Green's functions.

As the beam-column deflections become larger, the induced geometric nonlinearities result in effects that are not observed in linear systems. Contrary to the significant amount of attention in the literature concerning the linear dynamic analysis of beam-columns supported on elastic foundations, very little work has been done on the corresponding nonlinear problem. Lewandowski (1989) studied the nonlinear free vibration analysis of multispan beams on elastic supports, employing the dynamic FEM, neglecting horizontal and rotary inertia forces and considering the beams as distributed mass systems. Moreover, Arboleda-Monsalve et al. (2008) presented a Timoshenko beam-column resting on a two-parameter elastic foundation with generalized end conditions. The proposed model includes the frequency effects on the stiffness matrix and load vector as well as the coupling effects of bending and shear deformations along the member and the shear forces along the span induced by the applied axial load as the beam deforms according to the 'modified shear equation proposed by Timoshenko.

In this paper, a BEM is developed for the nonlinear analysis of shear deformable beam-columns of an arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on a nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary-value problems are formulated with respect to the transverse displacements, axial displacement, and two stress functions, and solved using the analog equation method (Katsikadelis 2002), which is a boundary element−based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

1. The shear deformation effect and rotary inertia are taken into account in the nonlinear dynamic analysis of beam-columns subjected to arbitrary loading (distributed or concentrated transverse loading and bending moments in both directions, as well as axial loading).

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2. The homogeneous linear half-space is approximated by a three-parameter viscoelastic foundation.
3. The beam-column is supported by the most general boundary conditions including elastic support or restrain, while its cross section is an arbitrary doubly symmetric one.
4. The proposed model takes into account the coupling effects of the bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading.
5. The shear deformation coefficients are evaluated using an energy approach, instead of the Timoshenko and Goodier (1984) and Cowper (1966) definitions, for which several authors (Schramm et al. 1994, 1997) have pointed out obtain unsatisfactory results, or instead of definitions given by other researchers (Stephen 1980; Hutchinson 2001) for which these factors take negative values.
6. The effect of the material's Poisson ratio, ν, is taken into account.
7. The proposed method employs a BEM approach (requiring boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Statement of the Problem

A prismatic beam-column of length l of a constant arbitrary doubly symmetric cross section of area A is considered (Fig. 1). The homogeneous isotropic and linearly elastic material of the beam-column cross section, with modulus of elasticity E, shear modulus G, and Poisson's ratio v, occupies the 2D multiply connected region Ω of the y,z plane and is bounded by the Γ_j ($j = 1,2,\ldots,K$) boundary curves, which are piecewise smooth; i.e., they may have a finite number of corners. In Fig. 1(b) Cyz is the principal bending coordinate system through the cross section's centroid. The beam-column is partially supported on a homogeneous nonlinear three-parameter viscoelastic soil. The foundation model is characterized by linear Winkler moduli k_{Ly} and k_{Lz}, nonlinear Winkler moduli k_{NLy} and k_{NLz}, Pasternak (shear) foundation moduli k_{Py} and k_{Pz}, and the damping coefficients c_y and c_z corresponding to the directions y and z, respectively. Thus, the foundation reaction is written as

$$p_{sy}(x,t) = k_{Ly}v(x,t) + k_{NL}v^3(x,t) - k_{Py}\frac{\partial^2 v(x,t)}{\partial x^2} + c_y\frac{\partial v(x,t)}{\partial t} \quad (1a)$$

$$p_{sz}(x,t) = k_{Lz}w(x,t) + k_{NL}w^3(x,t) - k_{Pz}\frac{\partial^2 w(x,t)}{\partial x^2} + c_z\frac{\partial w(x,t)}{\partial t} \quad (1b)$$

The beam-column is subjected to the combined action of the arbitrarily distributed or concentrated time-dependent axial loading $p_x = p_x(x,t)$; transverse loading $p_y = p_y(x,t)$ and $p_z = p_z(x,t)$ acting in the y and z directions, respectively; and bending moments $m_y = m_y(x,t)$ and $m_z = m_z(x,t)$ along the y and z axes, respectively [Fig. 1(a)].

Under the action of the aforementioned loading, the displacement field of the beam taking into account the shear deformation effect is given as

$$\bar{u}(x,y,z,t) = u(x,t) - y\theta_z(x,t) + z\theta_y(x,t) \quad (2a)$$

$$\bar{v}(x,t) = v(x,t) \quad (2b)$$

$$\bar{w}(x,t) = w(x,t) \quad (2c)$$

where $\bar{u}$, $\bar{v}$, and $\bar{w}$ = axial and transverse beam displacement components with respect to the the Cyz system of axes; $u(x,t)$, $v(x,t)$, and $w(x,t)$ = corresponding components of the centroid C; and $\theta_y(x,t)$ and $\theta_z(x,t)$ = angles of rotation as a result of bending of the cross section with respect to its centroid.

Employing the strain-displacement relationships of the three-dimensional elasticity for moderate displacements (Ramm and Hofmann 1995; Rothert and Gensichen 1987), the following strain components can be easily obtained:

$$\varepsilon_{xx} = \frac{\partial\bar{u}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial\bar{v}}{\partial x}\right)^2 + \left(\frac{\partial\bar{w}}{\partial x}\right)^2\right] \quad (3a)$$

$$\gamma_{xz} = \frac{\partial\bar{w}}{\partial x} + \frac{\partial\bar{u}}{\partial z} + \left(\frac{\partial\bar{v}}{\partial x}\frac{\partial\bar{v}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\frac{\partial\bar{w}}{\partial z}\right) \quad (3b)$$

$$\gamma_{xy} = \frac{\partial\bar{v}}{\partial x} + \frac{\partial\bar{u}}{\partial y} + \left(\frac{\partial\bar{v}}{\partial x}\frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial x}\frac{\partial\bar{w}}{\partial y}\right) \quad (3c)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \quad (3d)$$

where it has been assumed that for moderate displacements $(\partial\bar{u}/\partial x)^2 \ll \partial\bar{u}/\partial x$, $(\partial\bar{u}/\partial x)(\partial\bar{u}/\partial z) \ll (\partial\bar{w}/\partial x) + (\partial\bar{u}/\partial z)$, and $(\partial\bar{u}/\partial x)(\partial\bar{u}/\partial y) \ll (\partial\bar{v}/\partial x) + (\partial\bar{u}/\partial y)$. Substituting the displacement

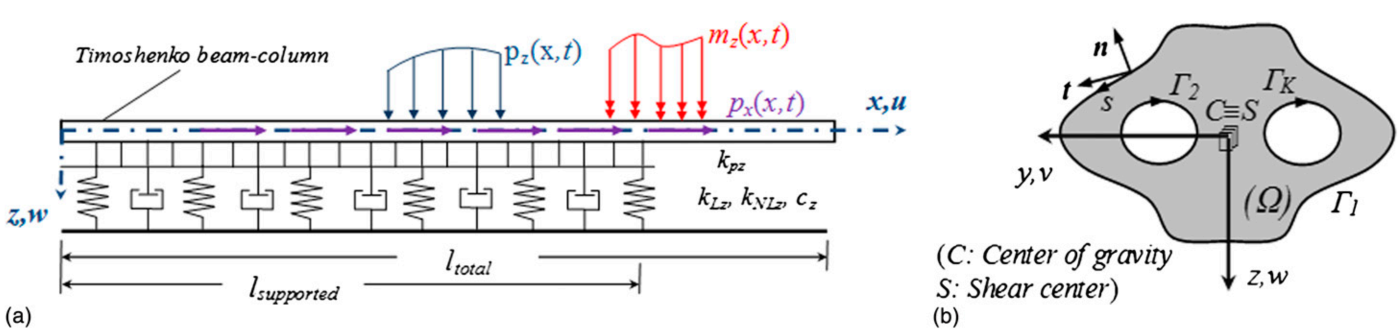

Fig. 1. (a) x-z plane of a prismatic beam-column partially resting on a three-parameter nonlinear viscoelastic foundation with (b) an arbitrary cross section occupying 2D region Ω

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components [Eqs. (2a)–(2c)] to the strain-displacement relationships [Eqs. (3a)–(3d)], the strain components can be written as

$$\varepsilon_{xx}(x,y,z,t) = u' + z\theta'_y - y\theta'_z + \frac{1}{2}\left(v'^2 + w'^2\right) \tag{4a}$$

$$\gamma_{xy} = v' - \theta_z \tag{4b}$$

$$\gamma_{xz} = w' + \theta_y \tag{4c}$$

where γ_{xy} and γ_{xz} = additional angles of rotation of the cross section as a result of shear deformation.

Considering strains to be small, employing the second Piola-Kirchhoff stress tensor and assuming an isotropic and homogeneous material, the stress components are defined in terms of the strain ones as

$$\begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \tag{5}$$

or employing Eqs. (4a)–(4c) as

$$S_{xx} = E\left[u' + z\theta'_y - y\theta'_z + \frac{1}{2}\left(v'^2 + w'^2\right)\right] \tag{6a}$$

$$S_{xy} = G\cdot\left(v' - \theta_z\right) \tag{6b}$$

$$S_{xz} = G\cdot\left(w' + \theta_y\right) \tag{6c}$$

On the basis of Hamilton's principle, the variations of the Lagrangian equation defined as

$$\delta\int_{t_1}^{t_2}(U - K - W_{\text{ext}})\,dt = 0 \tag{7}$$

and expressed as a function of the stress resultants acting on the cross section of the beam-column in the deformed state provide the governing equations and the boundary conditions of the beam-column subjected to nonlinear vibrations. In Eq. (7), $\delta(\cdot)$ denotes the variation of quantities while U, K, and W_{ext} are the strain energy, kinetic energy, and external load work, respectively, given as

$$\delta U = \int_V\left(S_{xx}\,\delta\varepsilon_{xx} + S_{xy}\,\delta\gamma_{xy} + S_{xz}\,\delta\gamma_{xz}\right)dV \tag{8a}$$

$$\delta K = \frac{1}{2}\int_V \rho\left(\delta\dot{u}^2 + \delta\dot{v}^2 + \delta\dot{w}^2\right)dV \tag{8b}$$

$$\delta W_{\text{ext}} = \int_L\Big[p_x\delta u + p_y\delta v + p_z\delta w + m_y\delta\theta_y + m_z\delta\theta_z - \delta\left(p_{sy}v\right) - \delta\left(p_{sz}w\right)\Big]dx \tag{8c}$$

Moreover, the stress resultants of the beam are given as

$$N = \int_\Omega S_{xx}d\Omega \tag{9a}$$

$$M_y = \int_\Omega S_{xx}z\,d\Omega \tag{9b}$$

$$M_z = -\int_\Omega S_{xx}y\,d\Omega \tag{9c}$$

$$Q_y = \int_{A_y} S_{xy}d\Omega \tag{9d}$$

$$Q_z = \int_{A_z} S_{xz}d\Omega \tag{9e}$$

Substituting the expressions of the stress components [Eqs. (6a)–(6c)] into Eqs. (9a)–(9e), the stress resultants are obtained as

$$N = EA\left[u' + \frac{1}{2}\left(v'^2 + w'^2\right)\right] \tag{10a}$$

$$M_y = EI_y\theta'_y \tag{10b}$$

$$M_z = EI_z\theta'_z \tag{10c}$$

$$Q_y = GA_y\gamma_{xy} \tag{10d}$$

$$Q_z = GA_z\gamma_{xz} \tag{10e}$$

where A = cross-sectional area and I_y and I_z = moments of inertia with respect to the principle bending axes given as

$$A = \int_\Omega d\Omega \tag{11}$$

$$I_y = \int_\Omega z^2 d\Omega \tag{12a}$$

$$I_z = \int_\Omega y^2 d\Omega \tag{12b}$$

and GA_y and GA_z = shear rigidities of the Timoshenko beam theory, where

$$A_z = \kappa_z A = \frac{1}{a_z}A \tag{13a}$$

$$A_y = \kappa_y A = \frac{1}{a_y}A \tag{13b}$$

= shear areas with respect to the y and z axes, respectively; κ_y and κ_z = shear correction factors; and a_y and a_z = shear deformation coefficients. Substituting the stress components given in Eqs. (6a)–(6c) and the strain resultants given in Eqs. (4a)–(4c) with the strain energy variation δU [Eq. (8a)] and employing Eq. (7), the equilibrium equations of the beam are derived as

$$-EA\left(u'' + w'w'' + v'v''\right) + \rho A\ddot{u} = p_x \tag{14a}$$

$$-\left(Nv'\right)' + \rho A\ddot{v} - GA_y\left(v'' - \theta'_z\right) + p_{sy} = p_y \tag{14b}$$

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$$-EI_z\theta_z'' + \rho I_z\ddot{\theta}_z - GA_y(v' - \theta_z) = m_z \tag{14c}$$

$$-(Nw')' + \rho A\ddot{w} - GA_z(w'' + \theta_y') + p_{sz} = p_z \tag{14d}$$

$$-EI_y\theta_y'' + \rho I_y\ddot{\theta}_y + GA_z(w' + \theta_y) = m_y \tag{14e}$$

Combining Eqs. (14*b*) and (14*c*), and Eqs. (14*d*) and (14*e*), the following differential equations with respect to u, v, and w are derived as

$$-EA(u'' + w'w'' + v'v'') + \rho A\ddot{u} = p_x \tag{15a}$$

$$\begin{aligned} &EI_z v'''' + \rho A\ddot{v} + p_{sy} + \frac{EI_z}{GA_y}\left((Nv')''' - \rho A\frac{\partial^2\ddot{v}}{\partial x^2} - p_{sy}'' + p_y''\right) \\ &- (Nv')' - \rho I_z\frac{\partial^2\ddot{v}}{\partial x^2} - \frac{\rho I_z}{GA_y}\left(\frac{\partial^2 (Nv')'}{\partial t^2} - \rho A\ddddot{v} - \ddot{p}_{sy} + \ddot{p}_y\right) \\ &= p_y - m_z' \end{aligned} \tag{15b}$$

$$\begin{aligned} &EI_y w'''' + \rho A\ddot{w} + p_{sz} + \frac{EI_y}{GA_z}\left((Nw')''' - \rho A\frac{\partial^2\ddot{w}}{\partial x^2} - p_{sz}'' + p_z''\right) \\ &- (Nw')' - \rho I_z\frac{\partial^2\ddot{w}}{\partial x^2} - \frac{\rho I_y}{GA_z}\left(\frac{\partial^2 (Nw')'}{\partial t^2} - \rho A\ddddot{w} - \ddot{p}_{sz} + \ddot{p}_z\right) \\ &= p_z + m_y' \end{aligned} \tag{15c}$$

Eqs. (15*a*)–(15*c*) constitute the governing differential equations of a Timoshenko beam-column, partially supported on a nonlinear three-parameter viscoelastic foundation, and subjected to nonlinear vibrations caused by the combined action of time-dependent axial and transverse loading. These equations are also subjected to the pertinent boundary conditions of the problem, which are given as

$$a_1 u(x,t) + \alpha_2 N(x,t) = \alpha_3 \tag{16}$$

$$\beta_1 v(x,t) + \beta_2 V_y(x,t) = \beta_3 \tag{17a}$$

$$\overline{\beta}_1\theta_z(x,t) + \overline{\beta}_2 M_z(x,t) = \overline{\beta}_3 \tag{17b}$$

$$\gamma_1 w(x,t) + \gamma_2 V_z(x,t) = \gamma_3 \tag{18a}$$

$$\overline{\gamma}_1\theta_y(x,t) + \overline{\gamma}_2 M_y(x,t) = \overline{\gamma}_3 \tag{18b}$$

at the beam-column ends $x = 0$, l, together with the initial conditions

$$u(x,0) = \overline{u}_0(x) \tag{19a}$$

$$\dot{u}(x,0) = \dot{\overline{u}}_0(x) \tag{19b}$$

$$v(x,0) = \overline{v}_0(x) \tag{20a}$$

$$\dot{v}(x,0) = \dot{\overline{v}}_0(x) \tag{20b}$$

$$w(x,0) = \overline{w}_0(x) \tag{21a}$$

$$\dot{w}(x,0) = \dot{\overline{w}}_0(x) \tag{21b}$$

where $\overline{u}_0(x)$, $\overline{v}_0(x)$, $\overline{w}_0(x)$, $\dot{\overline{u}}_0(x)$, $\dot{\overline{v}}_0(x)$, and $\dot{\overline{w}}_0(x)$ = prescribed functions. In Eqs. (17*a*), (17*b*), (18*a*), and (18*b*) V_y, V_z, M_z, and M_y are the reactions and bending moments with respect to y and z, respectively, which together with the angles of rotation caused by bending θ_y and θ_z are given by the following relationships:

$$V_y = Nv' - EI_z v''' - \frac{EI_z}{GA_y}\left[(Nv')'' + p_y' - p_{sy}' - \rho A\frac{\partial\ddot{v}}{\partial x}\right] + \rho I_z\ddot{\theta}_z \tag{22a}$$

$$V_z = Nw' - EI_y w''' - \frac{EI_y}{GA_z}\left[(Nw')'' + p_z' - p_{sz}' - \rho A\frac{\partial\ddot{w}}{\partial x}\right] - \rho I_y\ddot{\theta}_y \tag{22b}$$

$$M_z = EI_z v'' + \frac{EI_z}{GA_y}\left[(Nv')' + p_y - p_{sy} - \rho A\ddot{v}\right] \tag{22c}$$

$$M_y = -EI_y w'' - \frac{EI_y}{GA_z}\left[(Nw')' + p_z - p_{sz} - \rho A\ddot{w}\right] \tag{22d}$$

$$\begin{aligned} \theta_y = &\frac{EI_y}{G^2A_z^2}\left(-p_z' + p_{sz}' - (Nw')'' + \rho A\frac{\partial\ddot{w}}{\partial x}\right) \\ &- \frac{1}{GA_z}\left(EI_y w''' + \rho I_y\ddot{\theta}_y + GA_z w'\right) \end{aligned} \tag{22e}$$

$$\begin{aligned} \theta_z = &\frac{EI_z}{G^2A_y^2}\left(p_y' - p_{sy}' + (Nv')'' - \rho A\frac{\partial\ddot{v}}{\partial x}\right) \\ &+ \frac{1}{GA_y}\left(EI_z v''' - \rho I_z\ddot{\theta}_z + GA_y v'\right) \end{aligned} \tag{22f}$$

Finally, $\alpha_k, \beta_k, \overline{\beta}_k, \gamma_k$, and $\overline{\gamma}_k$ (k = 1,2,3) = functions specified at the beam ends $x = 0,l$. Eqs. (16)–(18) describe the most general nonlinear boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of conventional boundary conditions (clamped, simply supported, free, or guided edge) can be derived from these equations by specifying appropriately these functions (e.g., for a clamped edge it is $\alpha_1 = \beta_1 = \gamma_1 = 1, \overline{\beta}_1 = \overline{\gamma}_1 = 1$, and $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \overline{\beta}_2 = \overline{\beta}_3 = \overline{\gamma}_2 = \overline{\gamma}_3 = 0$).

The solution to the initial boundary-value problem given in Eqs. (15*a*)–(15*c*), and subjected to boundary conditions (16)–(18) and initial conditions (19)–(21), which represent the nonlinear flexural dynamic analysis of a Timoshenko beam-column, partially supported on a nonlinear three-parameter viscoelastic foundation, presumes the evaluation of the shear deformation coefficients a_y and a_z, corresponding to the principal coordinate system Cyz. These coefficients are established equating the approximate formula of the shear strain energy per unit length (Stephen 1980)

$$U_{\text{approx}} = \frac{a_y Q_y^2}{2AG} + \frac{a_z Q_z^2}{2AG} \tag{23}$$

with the exact one given from

$$U_{\text{exact}} = \int_\Omega \frac{(\tau_{xz})^2 + (\tau_{xy})^2}{2G} d\Omega \tag{24}$$

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and are obtained as (Sapountzakis and Mokos 2005)

$$a_y = \frac{1}{\kappa_y} = \frac{A}{\Delta^2}\int_\Omega [(\nabla\Theta) - \boldsymbol{e}]\cdot[(\nabla\Theta) - \boldsymbol{e}]d\Omega \tag{25a}$$

$$a_z = \frac{1}{\kappa_z} = \frac{A}{\Delta^2}\int_\Omega [(\nabla\Phi) - \boldsymbol{d}]\cdot[(\nabla\Phi) - \boldsymbol{d}]d\Omega \tag{25b}$$

where $(\tau_{xz})_j$ and $(\tau_{xy})_j$ = transverse and direct shear stress components, respectively; $(\nabla) \equiv \mathbf{i}_y(\partial/\partial y) + \mathbf{i}_z(\partial/\partial z)$ = symbolic vector with $\mathbf{i}_y$ and $\mathbf{i}_z$ being the unit vectors along the y and z axes, respectively; and Δ is given from

$$\Delta = 2(1+\nu)I_y I_z \tag{26}$$

where ν = Poisson ratio of the cross-sectional material and $\mathbf{e}$ and $\mathbf{d}$ = vectors defined as

$$\mathbf{e} = \left(\nu I_y \frac{y^2 - z^2}{2}\right)\mathbf{i}_y + \nu I_y yz\mathbf{i}_z \tag{27a}$$

$$\mathbf{d} = \nu I_z yz\mathbf{i}_y - \left(\nu I_z \frac{y^2 - z^2}{2}\right)\mathbf{i}_z \tag{27b}$$

where $\Theta(y,z)$ and $\Phi(y,z)$ = stress functions, which are evaluated from the solution to the following Neumann-type boundary-value problems (Sapountzakis and Mokos 2005)

$$\nabla^2\Theta = -2I_y y \quad \text{in} \quad \Omega \tag{28a}$$

$$\frac{\partial\Theta}{\partial n} = \mathbf{n}\cdot\mathbf{e} \quad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{28b}$$

$$\nabla^2\Phi = -2I_z z \quad \text{in} \quad \Omega \tag{29a}$$

$$\frac{\partial\Phi}{\partial n} = \mathbf{n}\cdot\mathbf{d} \quad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{29b}$$

where $\mathbf{n}$ = outward normal vector to boundary Γ. In the case of negligible shear deformations $a_z = a_y = 0$. Here, boundary conditions (28b) and (29b) have been derived from the physical consideration that the traction vector in the direction of normal vector $\mathbf{n}$ vanishes on the free surface of the beam-column.

Integral Representations—Numerical Solution

According to the precedent analysis, the nonlinear flexural dynamic analysis of Timoshenko beam-columns, partially supported on a nonlinear three-parameter viscoelastic foundation, and undergoing moderate large deflections reduces in establishing the displacement components $u(x,t)$, $v(x,t)$, and $w(x,t)$ having continuous derivatives up to second order and up to fourth order with respect to x, respectively, and also having derivatives up to second order with respect to t [ignoring the inertia terms of the fourth order (Thomson 1981)]. Moreover, these displacement components must satisfy coupled governing differential Eqs. (15a)–(15c) inside the beam-column, boundary conditions (16)–(18) at the beam ends $x = 0,l$, and initial conditions (19)–(21). Eqs. (15a)–(15c) are solved using the analog equation method (Katsikadelis 2002) as it is developed for hyperbolic differential equations (Sapountzakis and Katsikadelis 2000)

Transverse Displacements v and w

Let $v(x,t)$ and $w(x,t)$ be the solution sought for the aforementioned boundary-value problem. Setting $u_2(x,t) = v(x,t)$ and $u_3(x,t) = w(x,t)$, and differentiating these functions four times with respect to x, yields

$$\frac{\partial^4 u_i}{\partial x^4} = q_i(x,t)(i = 2,3) \tag{30}$$

Eq. (30) is quasi-static; that is, the time variable appears as a parameter. It indicates that the solution to Eqs. (15b) and (15c) can be established by solving Eq. (30) under the same boundary conditions (17) and (18), provided that the fictitious load distributions $q_i(x,t)$ (i = 2,3) are first established. (It refers to Eq. (30)). These distributions can be determined using the BEM as follows.

The solution to Eq. (30) is given in integral form as

$$u_i(x,t) = \int_0^l q_i(\xi,t)u^* d\xi - \left[u^*\frac{\partial^3 u_i}{\partial x^3} - \frac{du^*}{dx}\frac{\partial^2 u_i}{\partial x^2} + \frac{d^2u^*}{dx^2}\frac{\partial u_i}{\partial x} - \frac{d^3u^*}{dx^3}u_i\right]_0^l \tag{31}$$

where u^* = fundamental solution given as

$$u^* = \frac{1}{12}l^3\left(2 + \left|\frac{r}{l}\right|^3 - 3\left|\frac{r}{l}\right|^2\right) \tag{32}$$

where $r = x - \xi$ and x and ξ = points of the beam, which is a particular singular solution to the equation

$$\frac{d^4u^*}{dx^4} = \delta(x - \xi) \tag{33}$$

Employing Eq. (32), integral representation (31) can be written as

$$u_i(x,t) = \int_0^l q_i(\xi,t)\Lambda_4(r)d\xi - \left[\Lambda_4(r)\frac{\partial^3 u_i}{\partial x^3} + \Lambda_3(r)\frac{\partial^2 u_i}{\partial x^2} + \Lambda_2(r)\frac{\partial u_i}{\partial x} + \Lambda_1(r)u_i\right]_0^l \tag{34}$$

where kernels $\Lambda_j(r)$ (j = 1,2,3,4) are given as

$$\Lambda_1(r) = -\frac{1}{2}\,\mathrm{sgn}\,\frac{r}{l} \tag{35a}$$

$$\Lambda_2(r) = -\frac{1}{2}\,l\left(1 - \left|\frac{r}{l}\right|\right) \tag{35b}$$

$$\Lambda_3(r) = -\frac{1}{4}\,l^2\left|\frac{r}{l}\right|\left(\left|\frac{r}{l}\right| - 2\right)\mathrm{sgn}\,\frac{r}{l} \tag{35c}$$

$$\Lambda_4(r) = \frac{1}{12}\,l^3\left(2 + \left|\frac{r}{l}\right|^3 - 3\left|\frac{r}{l}\right|^2\right) \tag{35d}$$

In Eq. (34) for the line integral $r = x - \xi$ and the x and ξ points are inside the beam, whereas for the rest of the terms $r = x - \zeta$, x is inside the beam, and ζ is at the beam ends $0,l$.

Differentiation of Eq. (34) with respect to x results in the integral representations of the derivatives of u_i (i = 2,3). Following the procedure presented in Sapountzakis and Katsikadelis (2000) and employing the constant element assumption for load distributions q_i along the L internal beam-column elements (because the numerical implementation becomes very simple and the results obtained are of high accuracy), the integral representations of displacement components u_i ($i = 2,3$) and their first derivatives with respect to x when applied to the beam ends ($0,l$), together with boundary conditions (17) and (18) are employed to express the unknown boundary quantities $u_i(\zeta,t)$, $u_{i,x}(\zeta,t)$, $u_{i,xx}(\zeta,t)$, and $u_{i,xxx}(\zeta,t)$ ($\zeta = 0,l$) in terms of q_i as

$$\begin{bmatrix} \mathbf{D}_{11} & \mathbf{0} & \mathbf{D}_{13} & \mathbf{D}_{14} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} & \mathbf{D}_{24} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{E}_{41} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_{2,xxx} \\ \hat{\mathbf{u}}_{2,xx} \\ \hat{\mathbf{u}}_{2,x} \\ \hat{\mathbf{u}}_{2} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\beta}_3 \\ \overline{\boldsymbol{\beta}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{Bmatrix} \mathbf{q}_2 \tag{36a}$$

$$\begin{bmatrix} \mathbf{G}_{11} & \mathbf{0} & \mathbf{G}_{13} & \mathbf{G}_{14} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} & \mathbf{G}_{24} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{E}_{41} & \mathbf{E}_{42} & \mathbf{E}_{43} & 0 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_{3,xxx} \\ \hat{\mathbf{u}}_{3,xx} \\ \hat{\mathbf{u}}_{3,x} \\ \hat{\mathbf{u}}_{3} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\gamma}_3 \\ \overline{\boldsymbol{\gamma}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{Bmatrix} \mathbf{q}_3 \tag{36b}$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{13}$, $\mathbf{D}_{14}$, $\mathbf{D}_{21}$, $\mathbf{D}_{22}$, $\mathbf{D}_{23}$, $\mathbf{D}_{24}$, $\mathbf{G}_{11}$, $\mathbf{G}_{13}$, $\mathbf{G}_{14}$, $\mathbf{G}_{21}$, $\mathbf{G}_{22}$, $\mathbf{G}_{23}$, and $\mathbf{G}_{24}$ = 2×2 known square matrices including the values of the functions $\beta_j, \overline{\beta}_j, \gamma_j$, and $\overline{\gamma}_j$ ($j = 1,2$) of Eqs. (17) and (18); $\boldsymbol{\beta}_3$, $\overline{\boldsymbol{\beta}}_3$, $\boldsymbol{\gamma}_3$, and $\overline{\boldsymbol{\gamma}}_3$ = 2×1 known column matrices including the boundary values of the functions $\beta_3, \overline{\beta}_3, \gamma_3$, and $\overline{\gamma}_3$ of Eqs. (17) and (18); $\mathbf{E}_{jk}$ ($j = 3,4$; $k = 1,2,3,4$) = square 2×2 known coefficient matrices; and $\mathbf{F}_j$ ($j = 3,4$) = $2 \times L$ rectangular known matrices originating from the integration of kernels on the axis of the beam-column. Moreover,

$$\hat{\mathbf{u}}_i = \{u_i(0,t) u_i(l,t)\}^T \tag{37a}$$

$$\hat{\mathbf{u}}_{i,x} = \left\{\frac{\partial u_i(0,t)}{\partial x} \frac{\partial u_i(l,t)}{\partial x}\right\}^T \tag{37b}$$

$$\hat{\mathbf{u}}_{i,xx} = \left\{\frac{\partial^2 u_i(0,t)}{\partial x^2} \frac{\partial^2 u_i(l,t)}{\partial x^2}\right\}^T \tag{37c}$$

$$\hat{\mathbf{u}}_{i,xxx} = \left\{\frac{\partial^3 u_i(0,t)}{\partial x^3} \frac{\partial^3 u_i(l,t)}{\partial x^3}\right\}^T \tag{37d}$$

= vectors including the two unknown boundary values of the respective boundary quantities and $\mathbf{q}_i = \{q_1^i q_2^i \ldots q_L^i\}^T$ ($i = 2,3$) = vector including the L unknown nodal values of the fictitious load.

Discretization of the integral representations of displacement components u_i ($i = 2,3$) and their derivatives with respect to x, and application to the L collocation nodal points yields

$$\mathbf{u}_i = \mathbf{T}_i \mathbf{q}_i + \mathbf{t}_i \quad i = 2,3 \tag{38a}$$

$$\mathbf{u}_{i,x} = \mathbf{T}_{ix} \mathbf{q}_i + \mathbf{t}_{ix} \quad i = 2,3 \tag{38b}$$

$$\mathbf{u}_{i,xx} = \mathbf{T}_{ixx} \mathbf{q}_i + \mathbf{t}_{ixx} \quad i = 2,3 \tag{38c}$$

$$\mathbf{u}_{i,xxx} = \mathbf{T}_{ixxx} \mathbf{q}_i + \mathbf{t}_{ixxx} \quad i = 2,3 \tag{38d}$$

$$\mathbf{u}_{i,xxxx} = \mathbf{q}_i \quad i = 2,3 \tag{38e}$$

where $\mathbf{u}_i$, $\mathbf{u}_{i,x}$, $\mathbf{u}_{i,xx}$, $\mathbf{u}_{i,xxx}$, and $\mathbf{u}_{i,xxxx}$ = vectors including the values of $u_i(x,t)$ and their derivatives at the L nodal points; $\mathbf{T}_i$, $\mathbf{T}_{ix}$, $\mathbf{T}_{ixx}$, and $\mathbf{T}_{ixxx}$ = known $L \times L$ matrices; and $\mathbf{t}_i$, $\mathbf{t}_{ix}$, $\mathbf{t}_{ixx}$, and $\mathbf{t}_{ixxx}$ = known $L \times 1$ matrices.

In the conventional BEM, the load vectors $\mathbf{q}_i$ are known and Eqs. (38a)–(38e) are used to evaluate $u_i(x,t)$ and their derivatives at the L nodal points. However, this cannot be done here because $\mathbf{q}_i$ are unknown. For this purpose, $2L$ additional equations are derived, which permit the establishment of $\mathbf{q}_i$. These equations result by applying Eqs. (15b) and (15c) to the L collocation points, which after ignoring the inertia terms of the fourth order arising from coupling of shear deformations and rotary inertia (Thomson 1981), lead to the formulation of the following set of $2L$ simultaneous equations

$$\mathbf{M}_2 \ddot{\mathbf{q}}_2 + \mathbf{S}_2 \dot{\mathbf{q}}_2 + \mathbf{f}_{s2}(\mathbf{q}_2) = \mathbf{f}_2 \tag{39a}$$

$$\mathbf{M}_3 \ddot{\mathbf{q}}_3 + \mathbf{S}_3 \dot{\mathbf{q}}_3 + \mathbf{f}_{s3}(\mathbf{q}_3) = \mathbf{f}_3 \tag{39b}$$

where matrices $\mathbf{M}_2$, $\mathbf{M}_3$, $\mathbf{S}_2$, and $\mathbf{S}_3$ ($L \times L$) and vectors $\mathbf{f}_{s2}$, $\mathbf{f}_{s3}$, $\mathbf{f}_2$, and $\mathbf{f}_3$ ($L \times 1$) are given as

$$\mathbf{M}_2 = \rho A \mathbf{T}_2 - \frac{\rho I_z}{G A_y}\left(\mathbf{N}_x \mathbf{T}_{2x} + \mathbf{N}\mathbf{T}_{2xx} - \overline{\mathbf{K}}_{Ly}\mathbf{T}_2 - \overline{\mathbf{K}}_{NLy}\mathbf{T}_2^3 + \overline{\mathbf{K}}_{Py}\mathbf{T}_{2xx}\right) - \rho I_z\left(\frac{E a_y}{G} + 1\right)\mathbf{T}_{2xx} \tag{40a}$$

$$\mathbf{S}_2 = \overline{\mathbf{C}}_y \mathbf{T}_2 - \frac{E I_z}{G A_y} \overline{\mathbf{C}}_y \mathbf{T}_{2xx} - \frac{2\rho I_z}{G A_y}(\mathbf{N}_{xt}\mathbf{T}_{2x} + \mathbf{N}_t \mathbf{T}_{2xx}) \tag{40b}$$

$$\begin{aligned} \mathbf{f}_{s2} = {} & E I_z\left[1 + \frac{1}{G A_y}\left(\overline{\mathbf{K}}_{py} + \mathbf{N}\right)\right]\mathbf{q}_2 - \frac{\rho I_z}{G A_y}[\mathbf{N}_{xtt}(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}) + \mathbf{N}_{tt}(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx})] \\ & + \overline{\mathbf{K}}_{Ly}(\mathbf{T}_2\mathbf{q}_2 + \mathbf{t}_2) + \overline{\mathbf{K}}_{NLy}(\mathbf{T}_2\mathbf{q}_2 + \mathbf{t}_2)^3 - \overline{\mathbf{K}}_{py}(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx}) - \mathbf{N}_x(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}) \\ & + \frac{E I_z}{G A_y}[\mathbf{N}_{xxx}(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}) + 3\mathbf{N}_{xx}(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx}) + 3\mathbf{N}_x(\mathbf{T}_{2xxx}\mathbf{q}_2 + \mathbf{t}_{2xxx})] \\ & - \frac{E I_z}{G A_y}\left[\overline{\mathbf{K}}_{Ly}(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx}) + \overline{\mathbf{K}}_{NLy}(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx})^3\right] - \mathbf{N}(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx}) \end{aligned} \tag{40c}$$

$$\mathbf{f}_2 = \mathbf{p}_y - \mathbf{m}_{z,x} - \frac{EI_z}{GA_y}(\mathbf{p}_{y,xx}) + \frac{\rho I_z}{GA_y}\mathbf{p}_{y,tt} \tag{40d}$$

$$\mathbf{M}_3 = \rho A\mathbf{T}_3 - \frac{\rho I_y}{GA_z}(\mathbf{N}_x\mathbf{T}_{3x} + \mathbf{N}\mathbf{T}_{3xx} - \overline{\mathbf{K}}_{\mathrm{L}z}\mathbf{T}_3 - \overline{\mathbf{K}}_{\mathrm{NL}z}\mathbf{T}_3^3 + \overline{\mathbf{K}}_{\mathrm{P}z}\mathbf{T}_{3xx}) - \rho \mathrm{I}_y\left(\frac{Ea_z}{G} + 1\right)\mathbf{T}_{3xx} \tag{40e}$$

$$\mathbf{S}_3 = \overline{\mathbf{C}}_z\mathbf{T}_3 - \frac{EI_y}{GA_z}\overline{\mathbf{C}}_z\mathbf{T}_{3xx} - \frac{2\rho I_y}{GA_z}(\mathbf{N}_{xt}\mathbf{T}_{3x} + \mathbf{N}_t\mathbf{T}_{3xx}) \tag{40f}$$

$$\begin{aligned}\mathbf{f}_\mathbf{s} = {} & EI_y\left[1 + \frac{1}{GA_z}(\overline{\mathbf{K}}_{pz} + \mathbf{N})\right]\mathbf{q}_3 - \frac{\rho I_y}{GA_z}[\mathbf{N}_{xtt}(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}) + \mathbf{N}_{tt}(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx})] \\ & + \overline{\mathbf{K}}_{Lz}(\mathbf{T}_3\mathbf{q}_3 + \mathbf{t}_3) + \overline{\mathbf{K}}_{NLz}(\mathbf{T}_3\mathbf{q}_3 + \mathbf{t}_3)^3 - \overline{\mathbf{K}}_{pz}(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx}) - \mathbf{N}_x(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}) \\ & + \frac{EI_y}{GA_z}[\mathbf{N}_{xxx}(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}) + 3\mathbf{N}_{xx}(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx}) + 3\mathbf{N}_x(\mathbf{T}_{3xxx}\mathbf{q}_3 + \mathbf{t}_{3xxx})] \\ & - \frac{EI_y}{GA_z}\left[\overline{\mathbf{K}}_{Lz}(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx}) + \overline{\mathbf{K}}_{NLz}(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx})^3\right] - \mathbf{N}(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx})\end{aligned} \tag{40g}$$

$$\mathbf{f}_3 = \mathbf{p}_z + \mathbf{m}_{y,x} - \frac{EI_y}{GA_z}(\mathbf{p}_{z,xx}) + \frac{\rho I_y}{GA_z}\mathbf{p}_{z,tt} \tag{40h}$$

where $\mathbf{N}$ and $\mathbf{N}_{km}$ $(k,\ m = x,\ t) = L \times L$ diagonal matrices containing the values of the axial force and its derivatives with respect to the k and m parameters at the L nodal points; $\mathbf{p}_y$, $\mathbf{p}_{y,xx}$, $\mathbf{p}_{y,tt}$, $\mathbf{p}_z$, $\mathbf{p}_{z,xx}$, $\mathbf{p}_{z,tt}$, $\mathbf{m}_{y,x}$, and $\mathbf{m}_{z,x} = L \times 1$ vectors containing the values of the external loading and its derivatives at these points; while $\overline{\mathbf{K}}_{\mathrm{L}y}$, $\overline{\mathbf{K}}_{\mathrm{L}z}$, $\overline{\mathbf{K}}_{\mathrm{NL}y}$, $\overline{\mathbf{K}}_{\mathrm{NL}z}$, $\overline{\mathbf{K}}_{\mathrm{P}y}$, $\overline{\mathbf{K}}_{\mathrm{P}z}$, $\overline{\mathbf{C}}_y$, and $\overline{\mathbf{C}}_z$ = diagonal matrices whose diagonal elements are given as

$$(\overline{\mathbf{K}}_{\mathrm{L}y})_{i,i} = (k_{\mathrm{L}y})_i \tag{41a}$$

$$(\overline{\mathbf{K}}_{\mathrm{L}z})_{i,i} = (k_{\mathrm{L}z})_i \tag{41b}$$

$$(\overline{\mathbf{K}}_{\mathrm{NL}y})_{i,i} = (k_{\mathrm{NL}y})_i \tag{41c}$$

$$(\overline{\mathbf{K}}_{\mathrm{NL}z})_{i,i} = (k_{\mathrm{NL}z})_i \tag{41d}$$

$$(\overline{\mathbf{K}}_{\mathrm{P}y})_{i,i} = (k_{\mathrm{P}y})_i \tag{41e}$$

$$(\overline{\mathbf{K}}_{\mathrm{P}z})_{i,i} = (k_{\mathrm{P}z})_i \tag{41f}$$

$$(\overline{\mathbf{C}}_y)_{i,i} = (c_y)_i \tag{41g}$$

$$(\overline{\mathbf{C}}_z)_{i,i} = (c_z)_i \tag{41h}$$

taking the values of the corresponding subgrade moduli at the ith nodal point.

Axial Displacement u

Let $u_1 = u(x,t)$ be the solution sought for the boundary-value problem described by Eqs. (15a) and (16). Differentiating this function two times yields

$$\frac{\partial^2 u_1}{\partial x^2} = q_1(x,t) \tag{42}$$

Eq. (42) indicates that the solution to the original problem can be obtained as the axial displacement of a beam with unit axial rigidity subjected to an axial fictitious load $q_1(x,t)$ under the same boundary conditions. The fictitious load is unknown.

The solution to Eq. (42) and its derivative are given in integral form as

$$u(x,t) = \int_o^l u^* q_x(\xi,t)d\xi - \left[u^*\frac{\partial u}{\partial x} - \frac{du^*}{dx}u\right]_0^l \tag{43a}$$

$$\frac{\partial u(x,t)}{\partial x} = -\int_o^l \frac{du^*}{dx}q_x(\xi,t)d\xi - \left[-\frac{du^*}{dx}\frac{\partial u}{\partial x}\right]_0^l \tag{43b}$$

where u^* = fundamental solution, which is given as

$$u^* = \frac{1}{2}|r| \tag{44}$$

Following the same aforementioned procedure, the discretized counterpart of the integral representations of displacement component u_1 and its first derivative with respect to x [Eqs. (43)], when applied to all nodal points in the interior of the beam-column, yields

$$\mathbf{u}_1 = \mathbf{T}_1\mathbf{q}_1 + \mathbf{t}_1 \tag{45a}$$

$$\mathbf{u}_{1,x} = \mathbf{T}_{1x}\mathbf{q}_1 + \mathbf{t}_{1x} \tag{45b}$$

where $\mathbf{T}_1$ and $\mathbf{T}_{1x}$ = known $L \times L$ matrices, similar to those previously mentioned for displacements u_2 and u_3. Application of Eq. (15a) to the L collocation points, after employing Eqs. (38) and (45) leads to the formulation of the following system of L equations with respect to $\mathbf{q_1}$, $\mathbf{q_2}$, and $\mathbf{q_3}$ fictitious load vectors

$$\begin{aligned}(\boldsymbol{\rho}\mathbf{A}\mathbf{T}_1)\ddot{\mathbf{q}}_1 - \mathbf{EA}\mathbf{q_1} = {} & \mathbf{EA}[(\mathbf{T}_{2xx}\mathbf{q}_2 + \mathbf{t}_{2xx})]_{dg}(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}) \\ & + \mathbf{EA}[(\mathbf{T}_{3xx}\mathbf{q}_3 + \mathbf{t}_{3xx})]_{dg}(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}) + \mathbf{p_x}\end{aligned} \tag{46}$$

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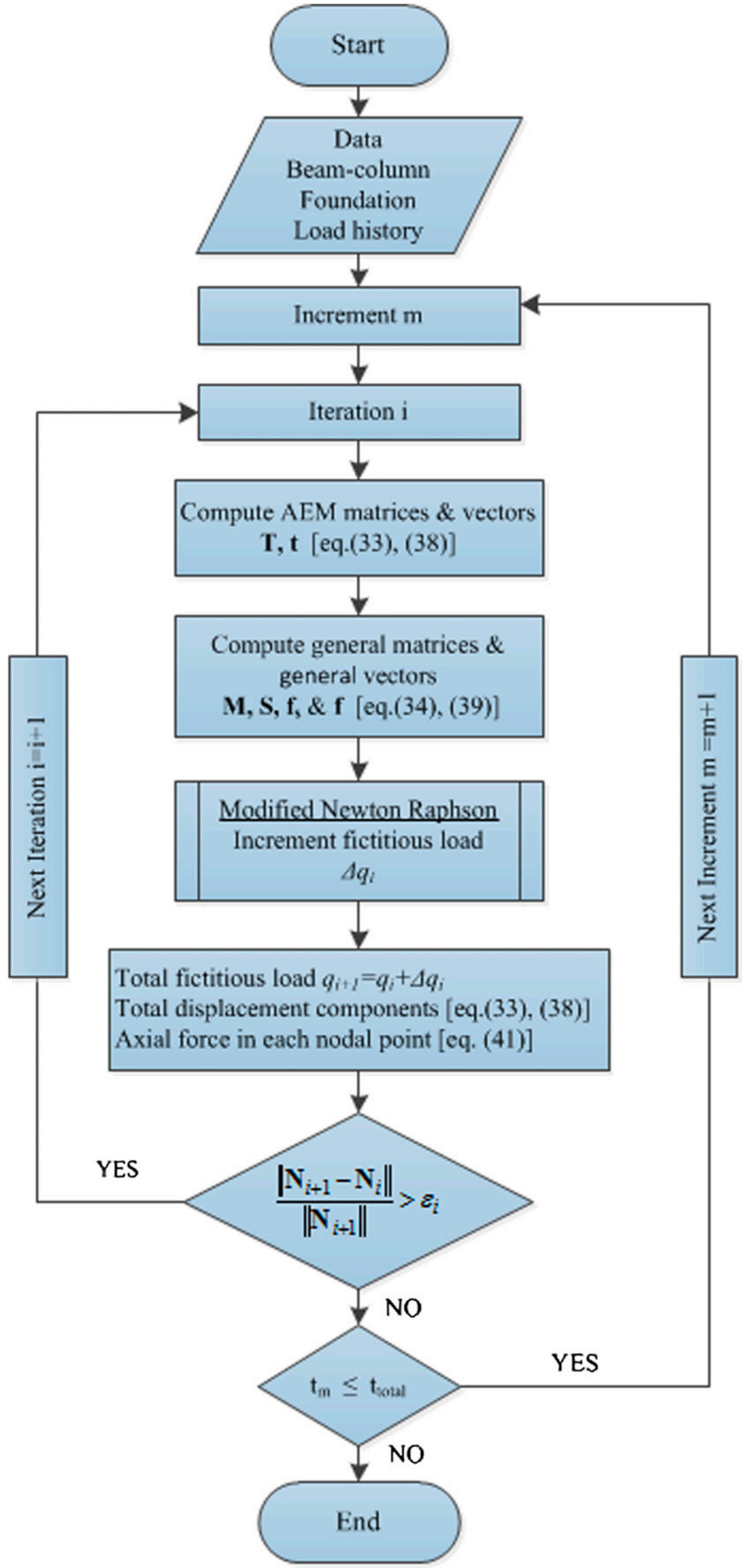

Fig. 2. Flowchart of the numerical implementation

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where $\mathbf{EA}$ and $\boldsymbol{\rho}\mathbf{A} = L \times L$ diagonal matrices including the values of the corresponding quantities at the L nodal points and $\overline{\mathbf{K}}_x$ is a diagonal matrix similar to those of Eqs. (41a)–(41h) whose diagonal elements are given as

$$(\overline{\mathbf{K}}_x)_{i,i} = (k_x)_i \tag{47}$$

Moreover, substituting Eqs. (38) and (46) in Eq. (10a) the discretized counterpart of the axial force at the neutral axis of the beam is given as

$$\mathbf{N} = \mathbf{EA}(\mathbf{T}_{1x}\mathbf{q}_1 + \mathbf{t}_{1x}) + \frac{1}{2}\mathbf{EA}[(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x})]_{dg}(\mathbf{T}_{2x}\mathbf{q}_2 + \mathbf{t}_{2x}) + \frac{1}{2}\mathbf{EA}[(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x})]_{dg}(\mathbf{T}_{3x}\mathbf{q}_3 + \mathbf{t}_{3x}) \tag{48}$$

Eqs. (39a), (39b), (46), and (48) constitute a nonlinear coupled system of equations with respect to $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, and $\mathbf{N}$ quantities. The solution to this system is accomplished iteratively by employing the average acceleration method in combination with the modified Newton-Raphson method (Chang 2004; Isaacson and Keller 1966). A step-by-step algorithmic approach of the numerical implementation is summarized in a flowchart form in Fig. 2.

Stress Functions $\Theta(y,z)$ and $\Phi(y,z)$

The evaluation of stress functions $\Theta(y,z)$ and $\Phi(y,z)$ is accomplished using the BEM as presented in Sapountzakis and Mokos (2005). Moreover, because the nonlinear flexural dynamic analysis of Timoshenko beam-columns, partially supported on a nonlinear three-parameter viscoelastic foundation, is solved by the BEM, the domain integrals for the evaluation of the area, bending moments of inertia [Eq. (11)], and shear deformation coefficients [Eqs. (25a) and (25b)] have to be converted to boundary line integrals to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem, and the Green identity. Thus, the moments, product of inertia, and cross-sectional area can be written as

$$I_y = \int_\Gamma (yz^2 n_y)\,ds \tag{49a}$$

$$I_z = \int_\Gamma (zy^2 n_z)\,ds \tag{49b}$$

$$A = \frac{1}{2}\int_\Gamma (yn_y + zn_z)\,ds \tag{49c}$$

while the shear deformation coefficients a_y and a_z are obtained from the relationships

$$a_y = \frac{A}{\Delta^2}\left[(4\nu + 2)I_y I_{\Theta y} + \frac{1}{4}\nu^2 I_{yy}^2 I_{ed} - I_{\Theta e}\right] \tag{50a}$$

$$a_z = \frac{A}{\Delta^2}\left[(4\nu + 2)I_z I_{\Phi z} + \frac{1}{4}\nu^2 I_z^2 I_{ed} - I_{\Phi d}\right] \tag{50b}$$

where

$$I_{\Theta e} = \int_\Gamma \Theta(\mathbf{n}\cdot\mathbf{e})\,ds \tag{51a}$$

$$I_{\Phi d} = \int_\Gamma \Phi(\mathbf{n}\cdot\mathbf{d})\,ds \tag{51b}$$

$$I_{ed} = \int_\Gamma \left(y^4 z n_z + z^4 y n_y + \frac{2}{3}y^2 z^3 n_z\right) ds \tag{51c}$$

$$I_{\Theta y} = \frac{1}{6}\int_\Gamma \left\{-2I_{yy}y^4 z n_z + [3\Theta n_y - y(\mathbf{n}\cdot\mathbf{e})]y^2\right\} ds \tag{51d}$$

$$I_{\Phi z} = \frac{1}{6}\int_\Gamma \left\{-2I_{zz}z^4 y n_y + [3\Phi n_z - z(\mathbf{n}\cdot\mathbf{d})]z^2\right\} ds \tag{51e}$$

Summary

In this paper, a boundary element method is developed for the nonlinear dynamic analysis of beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on a nonlinear three-parameter viscoelastic foundation, undergoing moderate large deflections under general boundary conditions, and taking into account the effects of shear deformation and rotary inertia. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions, as well as to axial loading. The proposed model takes into account the coupling effects of bending and shear deformations along the member, as well as the shear forces along the span induced by the applied axial loading. In Part II (Sapountzakis and Kampitsis 2013), the efficiency, accuracy, and range of applications for the proposed method will be discussed.

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