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E. J. Sapountzakis · A. E. Kampitsis

Nonlinear analysis of shear deformable beam-columns partially supported on tensionless three-parameter foundation

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Abstract In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross-section, partially supported on tensionless three-parameter foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to two stress functions and solved using the Analog Equation Method, a BEM-based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. Numerical examples are worked out to illustrate the efficiency, wherever possible, the accuracy and the range of applications of the developed method.

Keywords Nonlinear analysis · Large deflections · Timoshenko beam · Shear center · Shear deformation coefficients · Boundary element method · Nonlinear foundation · Tensionless foundation

1 Introduction

In the study of beams supported on elastic foundation, the bodies in contact (beam and subgrade) are not bonded to each other, and consequently, the admission of tensile stresses across the interface separating the beam from the foundation is not realistic. In this case, regions of no contact (which are unknown) develop beneath the beam and the change of the transverse displacement sign provides the condition for their determination.

Moreover, according to the modeling of the mechanical behavior of the subsoil and its interaction with the beam resting on it, assuming linear elastic, homogeneous, and isotropic soil behavior, the oldest, most famous and most frequently used mechanical model is the Winkler one. In this model, the supporting soil behavior is approximated by a series of closely spaced, mutually independent, linear elastic vertical springs, providing resistance in direct proportion to the deflection of the beam. However, this approximation due to its inability to take into account the continuity or cohesion of the soil (interaction between adjacent springs) is considered

E. J. Sapountzakis (✉) · A. E. Kampitsis
Institute of Structural Analysis and Antiseismic Research, School of Civil Engineering,
National Technical University of Athens, Zografou Campus, 15780 Athens, Greece
E-mail: cvsapoun@central.ntua.gr

A. E. Kampitsis
E-mail: cvakamb@gmail.com

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as a rather crude one. To overcome this weakness, a second parameter is introduced (Pasternak model) to account for the coupling effect of the linear elastic springs. The induction of this second parameter brings the modeling of the soil behavior closer to reality, but its response is still not as complicated as the elastic continuum model. This fact resulted in the development of more sophisticated models comprising three independent parameters for the description of the soil behavior. These three-parameter models constitute a generalization of two-parameter models, the third parameter being used to make them more realistic and effective. More specifically, since in practice the support structure may be highly nonlinear (e.g. hardening characteristics of the ballast and also the rail-pad), the inclusion of a third parameter associated with the cubic nonlinearity of the deflection renders the arising mechanical model capable of distributing stresses correctly [1].

Besides, the study of nonlinear effects on the analysis of structural elements is essential in civil engineering applications, wherein weight saving is of paramount importance. This nonlinearity results from retaining the square of the slope in the strain–displacement relations (intermediate nonlinear theory), avoiding in this way the inaccuracies arising from a linearized second-order analysis. Moreover, due to the intensive use of materials having relatively high transverse shear modulus, the error incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading.

Over the past thirty years, many researchers have developed and validated various methods of performing an analysis for beam-columns, partially supported on Winkler foundation, but only few of them took into account the realistic tensionless character of the subgrade reaction. To begin with, Sharma and Dasgupta [2] employed an iteration method using Green's functions for the analysis of uniformly loaded Bernoulli beams, followed by Kaschiev and Mikhajlov [3], who presented a finite element solution for beams subjected to arbitrary loading. Later Zhang and Murphy [4] presented for the same problem an analytical/numerical solution making no assumption about either the contact area or the kinematics associated with the transverse deflection of the beam. Avramidis and Morfidis [5] analyzed the bending problem of a Timoshenko beam on a Kerr-type three-parameter elastic foundation carrying out comparisons between one-, two-, or three-parameter foundation models. Maheshwari [6] employed the finite difference method with the help of appropriate boundary and continuity conditions for the analysis of beams on tensionless reinforced granular fill-soil system, while Ma et al. [7,8] used the transfer displacement function method (TDFM) to present the response of an infinite beam resting on a tensionless elastic foundation subjected to arbitrarily complex transverse loads. Zhang [9] analyzed a beam resting on a tensionless Reissner foundation and demonstrated the improvements in the Reissner foundation model compared with the Winkler one, while Ying et al. [10] presented exact solutions for bending and free vibration of functionally graded beams resting on a Winkler–Pasternak elastic foundation based on the two-dimensional theory of elasticity. Finally, Tullini and Tralli [11] presented a finite element solution for the static analysis of a foundation Timoshenko beam resting on elastic half-plane by employing locking-free Hermite polynomials. Nevertheless, in all of the aforementioned research efforts, only a geometrically linear analysis is performed.

As the deflections become larger, the induced geometric nonlinearities result in effects that are not observed in linear systems. Recently, Silveira et al. [12] presented a nonlinear analysis of Bernoulli structural elements under unilateral contact constraints employing a Ritz type approach, while Tsiatas [13] demonstrated a boundary integral equation solution to the nonlinear problem of nonuniform Bernoulli beams resting on a nonlinear triparametric elastic foundation. In these research efforts, the shear deformation effect is ignored.

In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross-section, partially supported on tensionless three-parameter foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, to the axial displacement, and to two stress functions and solved using the Analog Equation Method [14], a BEM-based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

(i) Shear deformation effect is taken into account on the geometrically nonlinear analysis of beam-columns subjected to arbitrary loading (distributed or concentrated transverse loading and bending moments in both directions, as well as axial loading).

(ii) The homogeneous linear half-space is approximated by a tensionless three-parameter foundation. The proposed method can also handle the case of negative foundation nonlinearity.
(iii) The beam-column is supported by the most general boundary conditions including elastic support or restraint, while its cross-section is an arbitrary doubly symmetric one.
(iv) The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading.
(v) The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's [15] and Cowper's [16] definitions, for which several authors [17,18] have pointed out that one obtains unsatisfactory results or definitions given by other researchers [19,20], for which these factors take negative values.
(vi) The effect of the material's Poisson's ratio ν is taken into account.
(vii) The proposed method employs a BEM approach (requiring boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross-section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples are worked out to illustrate the efficiency, wherever possible, the accuracy and the range of applications of the developed method.

2 Statement of the problem

Let us consider a prismatic beam-column of length l (Fig. 1), of constant arbitrary doubly symmetric cross-section of area A. The homogeneous isotropic and linear elastic material of the beam-column cross-section, with modulus of elasticity E, Poisson's ratio v, and shear modulus $G(G = E/(2(1+v)))$, occupies the two-dimensional multiply connected region Ω of the y, z plane and is bounded by the $\Gamma_j (j = 1, 2, \ldots, K)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1b, Cyz is the principal bending coordinate system through the cross-section's centroid. The beam-column is partially supported on a tensionless homogeneous elastic nonlinear three-parameter soil. The foundation model is characterized by the linear Winkler moduli k_x, k_{Ly}, k_{Lz}, the nonlinear Winkler moduli k_{NLy}, k_{NLz} and the Pasternak (shear) foundation moduli k_{Py}, k_{Pz} for the directions y, z, respectively. Having in mind that for the longitudinal direction, the reaction is a bilateral one exhibiting both positive and negative values, while for the transverse directions is a unilateral one (taking into account the unbonded contact between beam and

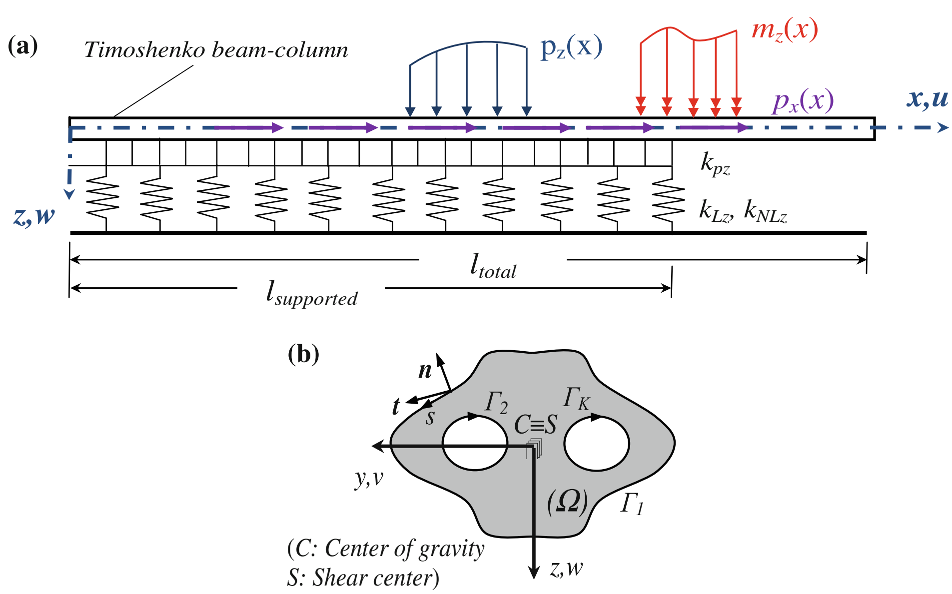

Fig. 1 x-z plane of a prismatic beam-column in axial–flexural loading (**a**) with an arbitrary doubly symmetric cross-section occupying the two-dimensional region Ω (**b**)

subgrade) developing only negative values, the interaction pressure at the interface can be written as

$$p_{sx} = k_x u(x) \tag{1a}$$

$$p_{sy} = U_v(x)\left(k_{Ly} v(x) + k_{NLy} v^3(x) - k_{Py} \frac{\partial^2 v(x)}{\partial x^2}\right) \tag{1b}$$

$$p_{sz} = U_w(x)\left(k_{Lz} w(x) + k_{NLz} w^3(x) - k_{Pz} \frac{\partial^2 w(x)}{\partial x^2}\right) \tag{1c}$$

where $U_v(x)$, $U_w(x)$ are unit step functions defined as

$$U_v(x) = \begin{cases} 1 & \text{if } \left(k_{Ly} v(x) + k_{NLy} v^3(x) - k_{Py} \frac{\partial^2 v(x)}{\partial x^2}\right) > 0 \\ 0 & \text{if } \left(k_{Ly} v(x) + k_{NLy} v^3(x) - k_{Py} \frac{\partial^2 v(x)}{\partial x^2}\right) \leq 0 \end{cases} \tag{2a}$$

$$U_w(x) = \begin{cases} 1 & \text{if } \left(k_{Lz} w(x) + k_{NLz} w^3(x) - k_{Pz} \frac{\partial^2 w(x)}{\partial x^2}\right) > 0 \\ 0 & \text{if } \left(k_{Lz} w(x) + k_{NLz} w^3(x) - k_{Pz} \frac{\partial^2 w(x)}{\partial x^2}\right) \leq 0 \end{cases} \tag{2b}$$

The beam is subjected to the combined action of the arbitrarily distributed or concentrated axial loading $p_x = p_x(x)$, transverse loading $p_y = p_y(x)$, $p_z = p_z(x)$ and bending moments $m_y = m_y(x)$, $m_z = m_z(x)$ acting along y, z directions, respectively (Fig. 1a).

Under the action of the aforementioned loading, the displacement field of the beam taking into account shear deformation effect is given as

$$\overline{u}(x, y, z) = u(x) - y\theta_z(x) + z\theta_y(x) \tag{3a}$$

$$\overline{v}(x) = v(x) \tag{3b}$$

$$\overline{w}(x) = w(x) \tag{3c}$$

where $\overline{u}$, $\overline{v}$, $\overline{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u(x)$, $v(x)$, $w(x)$ are the corresponding components of the centroid C and $\theta_y(x)$, $\theta_z(x)$ are the angles of rotation due to bending of the cross-section with respect to its centroid.

Employing the strain–displacement relations of the three-dimensional elasticity for moderate displacements [21,22], the following strain components can be easily obtained

$$\varepsilon_{xx} = \frac{\partial \overline{u}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \overline{v}}{\partial x}\right)^2 + \left(\frac{\partial \overline{w}}{\partial x}\right)^2\right] \tag{4a}$$

$$\gamma_{xz} = \frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{u}}{\partial z} + \left(\frac{\partial \overline{v}}{\partial x}\frac{\partial \overline{v}}{\partial z} + \frac{\partial \overline{w}}{\partial x}\frac{\partial \overline{w}}{\partial z}\right) \tag{4b}$$

$$\gamma_{xy} = \frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{u}}{\partial y} + \left(\frac{\partial \overline{v}}{\partial x}\frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial x}\frac{\partial \overline{w}}{\partial y}\right) \tag{4c}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \tag{4d}$$

where it has been assumed that for moderate displacements $(\partial\overline{u}/\partial x)^2 \ll \partial\overline{u}/\partial x$, $(\partial\overline{u}/\partial x)(\partial\overline{u}/\partial z) \ll (\partial\overline{w}/\partial x) + (\partial\overline{u}/\partial z)$, $(\partial\overline{u}/\partial x)(\partial\overline{u}/\partial y) \ll (\partial\overline{v}/\partial x) + (\partial\overline{u}/\partial y)$. Substituting the displacement components (3) to the strain–displacement relations (4), the strain components can be written as

$$\varepsilon_{xx}(x, y, z) = u' + z\theta_y' - y\theta_z' + \frac{1}{2}\left(v'^2 + w'^2\right) \tag{5a}$$

$$\gamma_{xy} = v' - \theta_z \tag{5b}$$

$$\gamma_{xz} = w' + \theta_y \tag{5c}$$

where γ_{xy}, γ_{xz} are the additional angles of rotation of the cross-section due to shear deformation.

Considering a beam-column element of length dx at its deformed shape and equating the external loads with the internal reaction, the equations of equilibrium are written as

$$\frac{\mathrm{d}N}{\mathrm{d}x} - k_x u + p_x = 0 \tag{6}$$

$$\frac{\mathrm{d}Q_y}{\mathrm{d}x} - p_{sy} + p_y = 0 \tag{7a}$$

$$\frac{\mathrm{d}Q_z}{\mathrm{d}x} - p_{sz} + p_z = 0 \tag{7b}$$

$$\frac{\mathrm{d}M_y}{\mathrm{d}x} - Q_z + m_y = 0 \tag{7c}$$

$$\frac{\mathrm{d}M_z}{\mathrm{d}x} + Q_y + m_z = 0 \tag{7d}$$

while the stress resultants after employing both the strain components of Eq. (5) and the stress–strain constitutive relations for an isotropic and homogeneous material are obtained as

$$N = EA\left[u' + \frac{1}{2}\left(v'^2 + w'^2\right)\right] \tag{8a}$$

$$M_y = EI_y\theta_y' \quad M_z = EI_z\theta_z' \tag{8b,c}$$

$$Q_y = GA_y\gamma_{xy} \quad Q_z = GA_z\gamma_{xz} \tag{8d,e}$$

where A is the cross-section area, I_y, I_z the moments of inertia with respect to the principle bending axes and GA_y, GA_z are its shear rigidities of the Timoshenko's beam theory, where

$$A_y = \kappa_y A = \frac{1}{a_y}A \quad A_z = \kappa_z A = \frac{1}{a_z}A \tag{9a,b}$$

are the shear areas with respect to y, z axes, respectively, with κ_y, κ_z the shear correction factors and a_y, a_z the shear deformation coefficients. Substituting the stress resultants of Eq. (8) and the strain resultants of Eq. (5) in the equilibrium equations (6), (7), the differential equations of equilibrium are written as

$$-EA\left(u'' + w'w'' + v'v''\right) + k_x u = p_x \tag{10a}$$

$$-\left(Nv'\right)' - GA_y\left(v'' - \theta_z'\right) + U_v\left(k_{Ly}v + k_{NLy}v^3 - k_{Py}v''\right) = p_y \tag{10b}$$

$$-EI_z\theta_z'' - GA_y\left(v' - \theta_z\right) = m_z \tag{10c}$$

$$-\left(Nw'\right)' - GA_z\left(w'' + \theta_y'\right) + U_w\left(k_{Lz}w + k_{NLz}w^3 - k_{Pz}w''\right) = p_z \tag{10d}$$

$$-EI_y\theta_y'' + GA_z\left(w' + \theta_y\right) = m_y \tag{10e}$$

Combining Eqs. (10b,c) and (10d,e), the governing differential equations with respect to u, v, w of a geometrically nonlinear Timoshenko beam-column, partially supported on a tensionless three-parameter foundation, subjected to the combined action of axial and transverse loading are obtained as

$$-EA\left(u'' + w'w'' + v'v''\right) + k_x u = p_x \tag{11a}$$

$$\begin{aligned} &EI_z v'''' + \left(\left(k_{Ly}v + k_{NLy}v^3 - k_{Py}v''\right) - \frac{EI_z}{GA_y}\left(k_{Ly}v + k_{NLy}v^3 - k_{Py}v''\right)''\right)U_v \\ &+\frac{EI_z}{GA_y}\left(Nv'\right)''' - \left(Nv'\right)' = p_y - \frac{EI_z}{GA_y}\left(p_y''\right) - m_z' \end{aligned} \tag{11b}$$

$$\begin{aligned} &EI_y w'''' + \left(\left(k_{Lz}w + k_{NLz}w^3 - k_{Pz}w''\right) - \frac{EI_y}{GA_z}\left(k_{Lz}w + k_{NLz}w^3 - k_{Pz}w''\right)''\right)U_w \\ &+\frac{EI_y}{GA_z}\left(Nw'\right)''' - \left(Nw'\right)' = p_z - \frac{EI_y}{GA_z}\left(p_z''\right) + m_y' \end{aligned} \tag{11c}$$

These equations are also subjected to the pertinent boundary conditions of the problem at hand, which are given as

$$a_1 u(x) + \alpha_2 N(x) = \alpha_3 \tag{12}$$

$$\beta_1 v(x) + \beta_2 V_y(x) = \beta_3 \quad \overline{\beta}_1 \theta_z(x) + \overline{\beta}_2 M_z(x) = \overline{\beta}_3 \tag{13a,b}$$

$$\gamma_1 w(x) + \gamma_2 V_z(x) = \gamma_3 \quad \overline{\gamma}_1 \theta_y(x) + \overline{\gamma}_2 M_y(x) = \overline{\gamma}_3 \tag{14a,b}$$

at the beam ends $x = 0, l$. In Eqs. (13a,b), (14a,b) V_y, V_z and M_y, M_z are the reactions and bending moments with respect to y, z axes, respectively, which together with the angles of rotation due to bending θ_y, θ_z are given as

$$V_y = -EI_z v''' - \frac{EI_z}{GA_y}\left[Nv''' - U_v\left(k_{Ly}v + k_{NLy}v^3 - k_{Py}v''\right)'\right] + Nv' \tag{15a}$$

$$V_z = -EI_y w''' - \frac{EI_y}{GA_z}\left[Nw''' - U_w\left(k_{Lz}w + k_{NLz}w^3 - k_{Pz}w''\right)'\right] + Nw' \tag{15b}$$

$$M_z = EI_z v'' + \frac{EI_z}{GA_y}\left[Nv'' - U_v\left(k_{Ly}v + k_{NLy}v^3 - k_{Py}v''\right)\right] \tag{15c}$$

$$M_y = -EI_y w'' - \frac{EI_y}{GA_z}\left[Nw'' - U_w\left(k_{Lz}w + k_{NLz}w^3 - k_{Pz}w''\right)\right] \tag{15d}$$

$$\theta_y = \frac{EI_y}{G^2A_z^2}\left(U_w\left(k_{Lz}w + k_{NLz}w^3 - k_{Pz}w''\right)' - \left(Nw'\right)''\right) - \frac{1}{GA_z}\left(EI_y w''' + GA_z w'\right) \tag{15e}$$

$$\theta_z = \frac{EI_z}{G^2A_y^2}\left(\left(Nv'\right)'' - U_v\left(k_{Ly}v + k_{NLy}v^3 - k_{Py}v''\right)'\right) + \frac{1}{GA_y}\left(EI_z v''' + GA_y v'\right) \tag{15f}$$

Finally, $\alpha_j, \beta_j, \overline{\beta}_j, \gamma_j, \overline{\gamma}_j (j = 1, 2, 3)$ are functions specified at the beam-column ends $x = 0, l$. Equations (12)–(14a,b) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge, it is $\alpha_1 = \beta_1 = \gamma_1 = 1, \overline{\beta}_1 = \overline{\gamma}_1 = 1, \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \overline{\beta}_2 = \overline{\beta}_3 = \overline{\gamma}_2 = \overline{\gamma}_3 = 0$).

The solution of the boundary value problem given from Eq. (11), subjected to the boundary conditions (Eqs. (12)–(14a,b)), which represents the nonlinear flexural analysis of a Timoshenko beam-column, partially supported on a tensionless three-parameter foundation, presumes the evaluation of the shear deformation coefficients a_y, a_z corresponding to the principal coordinate system Cyz. These coefficients are established equating the approximate formula of the shear strain energy per unit length [19]

$$U_{\text{appr.}} = \frac{a_y Q_y^2}{2AG} + \frac{a_z Q_z^2}{2AG} \tag{16}$$

with the exact one given from

$$U_{\text{exact}} = \int_{\Omega} \frac{(\tau_{xz})^2 + \left(\tau_{xy}\right)^2}{2G} \mathrm{d}\Omega \tag{17}$$

and are obtained as [23]

$$a_y = \frac{1}{\kappa_y} = \frac{A}{\Delta^2} \int_{\Omega} [(\nabla\Theta) - \boldsymbol{e}] \cdot [(\nabla\Theta) - \boldsymbol{e}]\, \mathrm{d}\Omega \tag{18a}$$

$$a_z = \frac{1}{\kappa_z} = \frac{A}{\Delta^2} \int_{\Omega} [(\nabla\Phi) - \boldsymbol{d}] \cdot [(\nabla\Phi) - \boldsymbol{d}]\, \mathrm{d}\Omega \tag{18b}$$

where $(\tau_{xz})_j$, $(\tau_{xy})_j$ are the transverse (direct) shear stress components, $(\nabla) \equiv \boldsymbol{i_y}(\partial/\partial y) + \boldsymbol{i_z}(\partial/\partial z)$ is a symbolic vector with $\boldsymbol{i_y}$, $\boldsymbol{i_z}$ the unit vectors along y and z axes, respectively, Δ is given from

$$\Delta = 2(1+\nu)I_y I_z \tag{19}$$

ν is the Poisson ratio of the cross-section material, $\boldsymbol{e}$ and $\boldsymbol{d}$ are vectors defined as

$$\boldsymbol{e} = \left(\nu I_y \frac{y^2 - z^2}{2}\right)\boldsymbol{i_y} + \nu I_y yz\boldsymbol{i}_z \tag{20a}$$

$$\boldsymbol{d} = \nu I_z yz\boldsymbol{i_y} - \left(\nu I_z \frac{y^2 - z^2}{2}\right)\boldsymbol{i}_z \tag{20b}$$

and $\Theta(y,z)$, $\Phi(y,z)$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems [23]

$$\nabla^2\Theta = -2I_y y \quad \text{in } \Omega \tag{21a}$$

$$\frac{\partial\Theta}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{e} \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{21b}$$

$$\nabla^2\Phi = -2I_z z \quad \text{in } \Omega \tag{22a}$$

$$\frac{\partial\Phi}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{d} \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{22b}$$

where $\boldsymbol{n}$ is the outward normal vector to the boundary Γ. In the case of negligible shear deformations, $a_y = a_z = 0$. It is also worth here noting that the boundary conditions (21b), (22b) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\boldsymbol{n}$ vanishes on the free surface of the beam.

3 Integral representations—numerical solution

According to the precedent analysis, the nonlinear flexural analysis of a Timoshenko beam-column, partially supported on a three-parameter foundation, undergoing moderate large deflections reduces in establishing the displacement components $u(x)$ and $v(x)$, $w(x)$ having continuous derivatives up to the second and up to the fourth order with respect to x, respectively. Moreover, these displacement components must satisfy the coupled governing differential equations (11) inside the beam and the boundary conditions (12)–(14a,b) at the beam ends $x = 0, l$. Equation (11) are solved using the Analog Equation Method [14], as it is developed for hyperbolic differential equations [24,25].

3.1 For the axial u and transverse displacements v, w

Let $u_1 = u(x)$ be the sought solution of the boundary value problem described by Eqs. (11a) and (12). Moreover, let $u_2(x) = v(x)$, $u_3(x) = w(x)$ be the sought solution of the boundary value problem described by Eqs. (11b,c) and (13a,b, 14a,b). Differentiating these functions two and four times with respect to x, respectively, yields

$$\frac{\partial^2 u_1}{\partial x^2} = q_1(x) \tag{23a}$$

$$\frac{\partial^4 u_i}{\partial x^4} = q_i(x) \quad (i = 2, 3) \tag{23b}$$

Equation (23) indicate that the solution of Eqs. (11a–c) can be established by solving Eq. (23) under the same boundary conditions (12–14a,b), provided that the fictitious. Elastic foundation reactions of Example 4 for various values of the factor γ. Load distributions $q_i(x)(i=1,2,3)$ are first established. These distributions can be determined using BEM as follows.

The solutions of Eq. (23) are given in integral form as

$$u_1(x,t)=\int_0^l q_1 u_1^* \mathrm{d}x-\left[u_1^*\frac{\partial u_1}{\partial x}-\frac{\mathrm{d}u_1^*}{\mathrm{d}x}u_1\right]_{x=0}^{x=l} \tag{24a}$$

$$u_i(x,t)=\int_0^l q_i u_i^* \mathrm{d}x-\left[u_i^*\frac{\partial^3 u_i}{\partial x^3}-\frac{\mathrm{d}u_i^*}{\mathrm{d}x}\frac{\partial^2 u_i}{\partial x^2}+\frac{\mathrm{d}^2u_i^*}{\mathrm{d}x^2}\frac{\partial u_i}{\partial x}-\frac{\mathrm{d}^3u_i^*}{\mathrm{d}x^3}u_i\right]_0^l \quad (i=2,3) \tag{24b}$$

where u_1^*, $u_i^*(i=2,3)$ are the fundamental solutions given as

$$u_1^*=\frac{1}{2}|r| \tag{25a}$$

$$u_i^*=\frac{1}{12}l^3\left(2+\left|\frac{r}{l}\right|^3-3\left|\frac{r}{l}\right|^2\right) \quad (i=2,3) \tag{25b}$$

with $r=x-\xi$, x,ξ points of the beam-column, which are particular singular solutions of the equations

$$\frac{\mathrm{d}^2u_1^*}{\mathrm{d}x^2}=\delta(x-\xi) \tag{26a}$$

$$\frac{\mathrm{d}^4u_i^*}{\mathrm{d}x^4}=\delta(x-\xi) \quad (i=2,3) \tag{26b}$$

Employing Eq. (25), the integral representations (24) can be written as

$$u_1(x,t)=\int_0^l q_1\left(\Lambda_2(r)+\frac{1}{2}l\right)\mathrm{d}x-\left[\left(\Lambda_2(r)+\frac{1}{2}l\right)\frac{\partial u_1}{\partial x}+\Lambda_1(r)u_1\right]_0^l \tag{27a}$$

$$u_i(x,t)=\int_0^l q_i\Lambda_4(r)\mathrm{d}x-\left[\Lambda_4(r)\frac{\partial^3 u_i}{\partial x^3}+\Lambda_3(r)\frac{\partial^2 u_i}{\partial x^2}+\Lambda_2(r)\frac{\partial u_i}{\partial x}+\Lambda_1(r)u_i\right]_0^l \tag{27b}$$

where the kernels $\Lambda_j(r)$, $(j=1,2,3,4)$ are given as

$$\Lambda_1(r)=-\frac{1}{2}sgn\frac{r}{l} \tag{28a}$$

$$\Lambda_2(r)=-\frac{1}{2}l\left(1-\left|\frac{r}{l}\right|\right) \tag{28b}$$

$$\Lambda_3(r)=-\frac{1}{4}l^2\left|\frac{r}{l}\right|\left(\left|\frac{r}{l}\right|-2\right)sgn\frac{r}{l} \tag{28c}$$

$$\Lambda_4(r)=\frac{1}{12}l^3\left(2+\left|\frac{r}{l}\right|^3-3\left|\frac{r}{l}\right|^2\right) \tag{28d}$$

It is worth here noting that in Eq. (27), for the line integrals it is $r=x-\xi$, x,ξ points inside the beam-column, whereas for the rest terms it is $r=x-\zeta$, x inside the beam-column, ζ at its end $0,l$.

Differentiating Eq. (27) with respect to x results in the integral representations of the derivatives of u_1 and u_i as

$$\frac{\partial u_1(x,t)}{\partial x} = \int_0^l q_1 \Lambda_1(r)\mathrm{d}x - \left[\Lambda_1(r)\frac{\partial u_1}{\partial x}\right]_{x=0}^{x=l} \tag{29a}$$

$$\frac{\partial^2 u_1(x,t)}{\partial x^2} = q_1(x,t) \tag{29b}$$

$$\frac{\partial u_i(x)}{\partial x} = \int_0^l q_i \Lambda_3(r)\mathrm{d}x - \left[\Lambda_3(r)\frac{\partial^3 u_i}{\partial x^3} + \Lambda_2(r)\frac{\partial^2 u_i}{\partial x^2} + \Lambda_1(r)\frac{\partial u_i}{\partial x}\right]_0^l \tag{29c}$$

$$\frac{\partial^2 u_i(x)}{\partial x^2} = \int_0^l q_i \Lambda_2(r)\mathrm{d}x - \left[\Lambda_2(r)\frac{\partial^3 u_i}{\partial x^3} + \Lambda_1(r)\frac{\partial^2 u_i}{\partial x^2}\right]_0^l \tag{29d}$$

$$\frac{\partial^3 u_i(x)}{\partial x^3} = \int_0^l q_i \Lambda_1(r)\mathrm{d}x - \left[\Lambda_1(r)\frac{\partial^3 u_i}{\partial x^3}\right]_0^l \tag{29e}$$

$$\frac{\partial^4 u_i(x,t)}{\partial x^4} = q_i(x,t) \tag{29f}$$

The integral representations (27a,b) and (29a,c) when applied to the beam-column ends $(0, l)$, together with the boundary conditions (12–14a,b), are employed to express the unknown boundary quantities $u_1(\zeta)$, $u_i(\zeta)(i = 2, 3)$ where $(\zeta = 0, l)$ and their derivatives in terms of q_i. This is accomplished numerically. More specifically, the interval $(0, l)$ is divided into L elements on which $q_i(x)(i = 1, 2, 3)$ are assumed to vary according to certain law (constant, linear, parabolic, etc.). The constant element assumption is employed here as the numerical implementation becomes very simple and the obtained results are very good.

Employing the aforementioned procedure, the following set of 20 nonlinear equations is obtained

$$\begin{bmatrix} \mathbf{T}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_{33} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{Bmatrix} + \begin{Bmatrix} \mathbf{D}_1^{\mathbf{nl}} \\ \mathbf{D}_2^{\mathbf{nl}} \\ \mathbf{D}_3^{\mathbf{nl}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{a}_3 \\ \boldsymbol{\beta}_3 \\ \boldsymbol{\gamma}_3 \end{Bmatrix} \tag{30}$$

with

$$\mathbf{T}_{11} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{0} & \mathbf{D}_{22} & \mathbf{D}_{23} \end{bmatrix} \tag{31a}$$

$$\mathbf{T}_{22} = \begin{bmatrix} \mathbf{F}_3 & \mathbf{E}_{35} & \mathbf{E}_{36} & \mathbf{E}_{37} & \mathbf{E}_{38} \\ \mathbf{F}_4 & \mathbf{0} & \mathbf{E}_{46} & \mathbf{E}_{47} & \mathbf{E}_{48} \\ \mathbf{0} & \mathbf{D}_{55} & \mathbf{D}_{56} & \mathbf{D}_{57} & \mathbf{D}_{58} \\ \mathbf{0} & \mathbf{D}_{65} & \mathbf{D}_{66} & \mathbf{0} & \mathbf{D}_{68} \end{bmatrix} \tag{31b}$$

$$\mathbf{T}_{33} = \begin{bmatrix} \mathbf{F}_7 & \mathbf{E}_{710} & \mathbf{E}_{711} & \mathbf{E}_{712} & \mathbf{E}_{713} \\ \mathbf{F}_8 & \mathbf{0} & \mathbf{E}_{811} & \mathbf{E}_{812} & \mathbf{E}_{813} \\ \mathbf{0} & \mathbf{D}_{910} & \mathbf{D}_{911} & \mathbf{D}_{912} & \mathbf{D}_{913} \\ \mathbf{0} & \mathbf{D}_{1010} & \mathbf{D}_{1011} & \mathbf{0} & \mathbf{D}_{1013} \end{bmatrix} \tag{31c}$$

where $\mathbf{D}_{22}$–$\mathbf{D}_{1013}$ are 2×2 rectangular known matrices including the values of the functions α_j, β_j, $\overline{\beta}_j$, γ_j, $\overline{\gamma}_j$ $(j = 1, 2)$ of Eqs. (17)–(19), $\mathbf{D}_1^{\mathbf{nl}}$, $\mathbf{a}_3$ and $\mathbf{D}_2^{\mathbf{nl}}$, $\mathbf{D}_3^{\mathbf{nl}}$, $\boldsymbol{\beta}_3$, $\boldsymbol{\gamma}_3$ are 4×1 and 8×1, respectively, known column matrices including the boundary values of the functions α_3, β_3, $\overline{\beta}_3$, γ_3, $\overline{\gamma}_3$ of Eqs. (12)–(14a,b). Moreover, $\mathbf{E}_{12}$–$\mathbf{E}_{813}$ are rectangular 2×2 known coefficient matrices resulting from the values of the kernels $\Lambda_j(r)(j = 1, 2, 3, 4)$ at the beam-column ends and $\mathbf{F}_1$–$\mathbf{F}_8$ are $2 \times L$ rectangular known matrices originating from the integration of the kernels on the axis of the beam-column, while the generalized unknown vectors

$\mathbf{d}_1^T = \left\{\mathbf{q}_1 \quad \hat{\mathbf{u}}_1 \quad \hat{\mathbf{u}}_{1,x}\right\}$, $\mathbf{d}_2^T = \left\{\mathbf{q}_2 \quad \hat{\mathbf{u}}_2 \quad \hat{\mathbf{u}}_{2,x} \quad \hat{\mathbf{u}}_{2,xx} \quad \hat{\mathbf{u}}_{2,xxx}\right\}$ and $\mathbf{d}_3^T = \left\{\mathbf{q}_3 \quad \hat{\mathbf{u}}_3 \quad \hat{\mathbf{u}}_{3,x} \quad \hat{\mathbf{u}}_{3,xx} \quad \hat{\mathbf{u}}_{3,xxx}\right\}$ contain the fictitious loads $q_i = \left\{q_1^i q_2^i \ldots q_L^i\right\}^T$ $(i = 1, 2, 3)$ and the vectors including the two unknown boundary values of the respective boundary quantities

$$\hat{\mathbf{u}}_i = \{u_i(0) \quad u_i(l)\}^T \quad (i = 1, 2, 3) \tag{32a}$$

$$\hat{\mathbf{u}}_{i,x} = \left\{\frac{\partial u_i(0)}{\partial x} \quad \frac{\partial u_i(l)}{\partial x}\right\}^T \quad (i = 1, 2, 3) \tag{32b}$$

$$\hat{\mathbf{u}}_{i,xx} = \left\{\frac{\partial^2 u_i(0)}{\partial x^2} \quad \frac{\partial^2 u_i(l)}{\partial x^2}\right\}^T \quad (i = 2, 3) \tag{32c}$$

$$\hat{\mathbf{u}}_{i,xxx} = \left\{\frac{\partial^3 u_i(0)}{\partial x^3} \quad \frac{\partial^3 u_i(l)}{\partial x^3}\right\}^T \quad (i = 2, 3) \tag{32d}$$

Discretization of Eqs. (27), (29) and application to the L collocation nodal points yields

$$\mathbf{u}_1 = \mathbf{H}_1^0\mathbf{d}_1 \tag{33a}$$

$$\mathbf{u}_{1,x} = \mathbf{H}_1^1\mathbf{d}_1 \tag{33b}$$

$$\mathbf{u}_i = \mathbf{H}_i^0\mathbf{d}_i \tag{33c}$$

$$\mathbf{u}_{i,x} = \mathbf{H}_i^1\mathbf{d}_i \tag{33d}$$

$$\mathbf{u}_{i,xx} = \mathbf{H}_i^2\mathbf{d}_i \tag{33e}$$

$$\mathbf{u}_{i,xxx} = \mathbf{H}_i^3\mathbf{d}_i \tag{33f}$$

where $\mathbf{H}_1^k, \mathbf{H}_i^j (i = 2, 3), (k = 0, 1), (j = 0, 1, 2, 3)$ are $L \times (L + 4)$ and $L \times (L + 8)$ known matrices. Applying Eq. (11) to the L collocation points and employing Eq. (33), $3L$ nonlinear equations are formulated as

$$\mathbf{k}^{nl}(\mathbf{d1}, \mathbf{d2}, \mathbf{d3}) = \mathbf{f} \tag{34}$$

where $\mathbf{k}^{nl}$ is a nonlinear generalized stiffness vector, and $\mathbf{f}$ is the generalized force vector. The above Eq. (34), together with Eq. (30), constitutes a system of $3L + 20$ nonlinear algebraic equations. The solution of this system is accomplished by using the modified Powell algorithm [26,27]. A step-by-step algorithmic approach of the nonlinear solution is presented in a flowchart form in Fig. 2.

3.2 For the stress functions $\Theta(y, z)$ and $\Phi(y, z)$

The evaluation of the stress functions $\Theta(y, z)$ and $\Phi(y, z)$ is accomplished using BEM as this is presented in Sapountzakis and Mokos [23].

Moreover, since the nonlinear flexural problem of Timoshenko beam-columns is solved by the BEM, the domain integrals for the evaluation of the area, the bending moments of inertia and the shear deformation coefficients (Eq. (18)) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments of inertia and the cross-section area can be written as

$$A = \frac{1}{2}\int_{\Gamma} (yn_y + zn_z)\mathrm{d}s \tag{35a}$$

$$I_y = \int_{\Gamma} \left(yz^2 n_y\right) \mathrm{d}s \quad I_z = \int_{\Gamma} \left(zy^2 n_z\right) \mathrm{d}s \tag{35b,c}$$

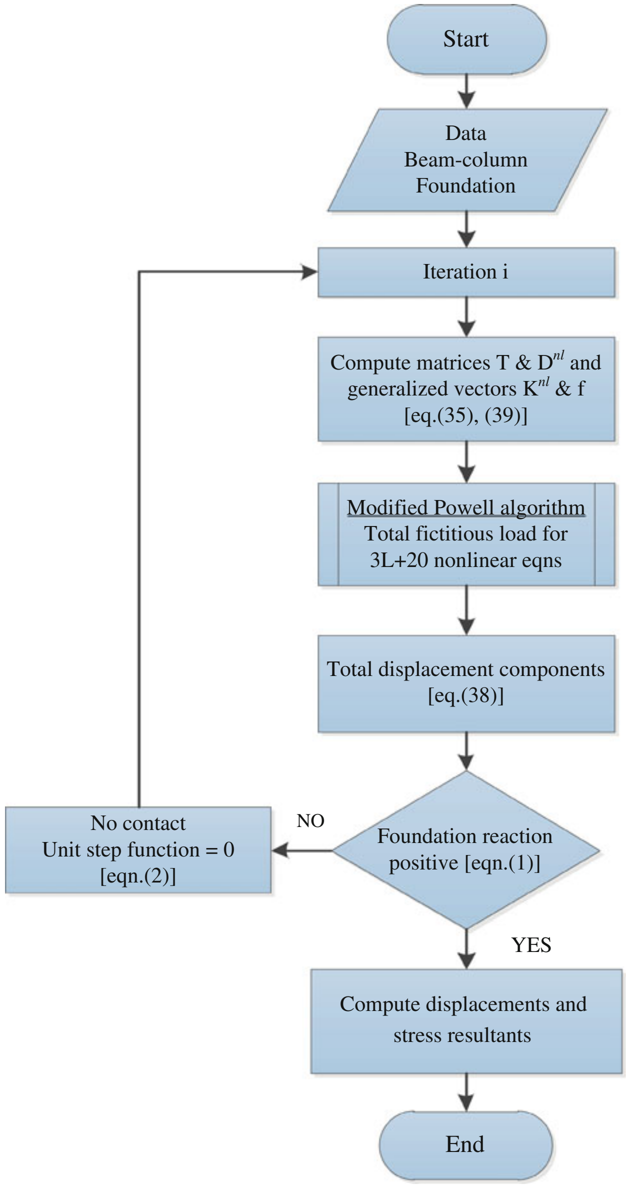

Fig. 2 Flowchart of the nonlinear algorithm

while the shear deformation coefficients a_y and a_z are obtained from the relations

$$a_y = \frac{A}{\Delta^2}\left((4v+2)\, I_y I_{\Theta y} + \frac{1}{4}v^2 I_{yy}^2 I_{ed} - I_{\Theta e}\right) \tag{36a}$$

$$a_z = \frac{A}{\Delta^2}\left((4v+2)\, I_z I_{\Phi z} + \frac{1}{4}v^2 I_z^2 I_{ed} - I_{\Phi d}\right) \tag{36b}$$

where

$$I_{\Theta e} = \int_{\Gamma} \Theta(\boldsymbol{n}\cdot\boldsymbol{e})\mathrm{d}s \tag{37a}$$

$$I_{\Phi d} = \int_{\Gamma} \Phi(\boldsymbol{n}\cdot\boldsymbol{d})\mathrm{d}s \tag{37b}$$

Table 1 Deflections (m) at the midpoint of the hinged-hinged beam-column of Example 1

k_{Lz} (kN/m^2)	k_{NLz} (kN/m^4)	k_{Pz} (kN)	With shear deformation	Without shear deformation	Tsiatas [13]
0	0	0	0.31189	0.31259	0.31272
1,000	0	0	0.25452	0.25537	0.25546
0	1,000	0	0.30742	0.30812	0.30825
0	0	1,000	0.25086	0.25172	0.25176
1,000	1,000	0	0.25168	0.25253	0.25262
1,000	0	1,000	0.20057	0.20152	0.20154
0	1,000	1,000	0.24820	0.24906	0.24909
1,000	1,000	1,000	0.19912	0.20005	0.20009

Table 2 Deflections (m) at the midpoint of the hinged-fixed beam-column of Example 1

k_{Lz} (kN/m^2)	k_{NLz} (kN/m^4)	k_{Pz} (kN)	With shear deformation	Without shear deformation	Tsiatas [13]
0	0	0	0.28398	0.28184	0.28207
1,000	0	0	0.23144	0.23022	0.23038
0	1,000	0	0.28052	0.27849	0.27871
0	0	1,000	0.22395	0.22222	0.22242
1,000	1,000	0	0.22933	0.22816	0.22832
1,000	0	1,000	0.17989	0.17894	0.17910
0	1,000	1,000	0.22204	0.22038	0.22058
1,000	1,000	1,000	0.17889	0.17796	0.17812

$$I_{ed} = \int_{\Gamma} \left(y^4 z n_z + z^4 y n_y + \frac{2}{3} y^2 z^3 n_z \right) \mathrm{d}s \tag{37c}$$

$$I_{\Theta y} = \frac{1}{6} \int_{\Gamma} \left[-2 I_{yy} y^4 z n_z + \left(3\Theta n_y - y(\boldsymbol{n} \cdot \boldsymbol{e}) \right) y^2 \right] \mathrm{d}s \tag{37d}$$

$$I_{\Phi z} = \frac{1}{6} \int_{\Gamma} \left[-2 I_{zz} z^4 y n_y + (3\Phi n_z - z(\boldsymbol{n} \cdot \boldsymbol{d}))\, z^2 \right] \mathrm{d}s \tag{37g}$$

4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible, the accuracy, and the range of applications of the developed method.

Example 1 For comparison reasons, a beam-column of length $l = 3$ m, resting on a three-parameter foundation ($E = 2.9 \times 10^6$ kN/m^2, $A = 0.02$ m^2, $I_y = 6.67 \times 10^{-5}$ m^4, $a_z = 1.19$) has been studied. The beam-column is subjected to a uniformly distributed load $p_z = 500$ kN/m, while three types of boundary conditions, namely (i) hinged-hinged, (ii) hinged-fixed, and (iii) fixed-fixed have been examined.

In Tables 1, 2 and 3, the central beam deflection for various values of the foundation parameters are presented taking into account or ignoring shear deformation effect compared with those obtained from a BEM solution ignoring this effect [13], for the aforementioned cases of boundary conditions, respectively. Moreover, in Fig. 3 the deflection w along the clamped beam-column resting on a three-parameter nonlinear foundation with $k_{Lz} = 1{,}000$ kN/m^2, $k_{NLz} = 1{,}000$ kN/m^4 and $k_{Pz} = 1{,}000$ kN is presented performing either a linear or a nonlinear analysis and taking into account or ignoring shear deformation effect. From this figure, the influence of the nonlinearity to the performed analysis is remarked.

Example 2 In order to illustrate the importance of the nonlinear analysis and the influence of the shear deformation effect, a cantilever beam-column of length $l = 1.0$ m, ($E = 21 \times 10^7$ kN/m^2, $v = 0.3$, $A = 2.9 \times 10^{-3}$ m^2, $I_y = 5.124 \times 10^{-6}$ m^4, $a_z = 4.513$) and resting on a Pasternak type foundation of stiffness $k_{Lz} = 2{,}000$ kN/m^2, $k_{Pz} = 1{,}000\, kN$, is examined. The beam-column is subjected to a uniformly distributed axial compressive

Table 3 Deflections (m) at the midpoint of the fixed-fixed beam-column of Example 1

k_{Lz} (kN/m^2)	k_{NLz} (kN/m^4)	k_{Pz} (kN)	With shear deformation	Without shear deformation	Tsiatas [13]
0	0	0	0.25747	0.25292	0.25324
1,000	0	0	0.20956	0.20651	0.20675
0	1,000	0	0.25492	0.25054	0.25086
0	0	1,000	0.19759	0.19359	0.19390
1,000	1,000	0	0.20807	0.20511	0.20533
1,000	0	1,000	0.15997	0.15757	0.15757
0	1,000	1,000	0.19633	0.19242	0.19274
1,000	1,000	1,000	0.15932	0.15673	0.15696

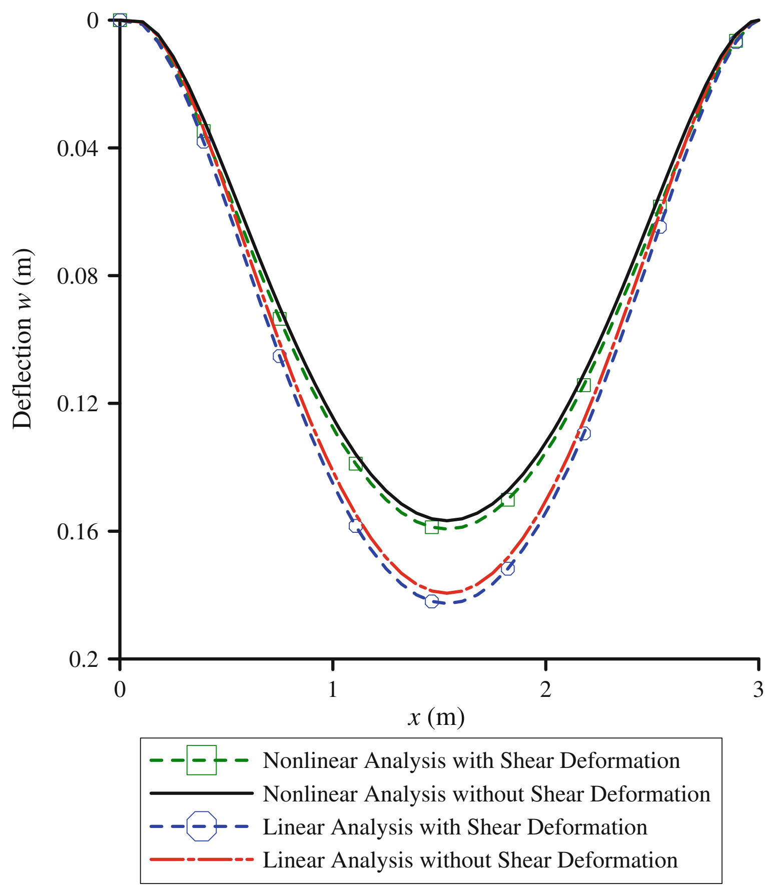

Fig. 3 Deflection w along the fixed-fixed beam-column of Example 1 for $k_{Lz} = 1{,}000\,\text{kN/m}^2$, $k_{NLz} = 1{,}000\,\text{kN/m}^4$ and $k_{Pz} = 1{,}000kN$

$p_x = 100\,\text{kN/m}$ and transverse $p_z = 200\,\text{kN/m}$ loading as well as to a concentrated compressive axial force at its end $P_x(l) = 200\,\text{kN}$.

In Fig. 4, the deflection curves w of the beam-column are presented performing either a linear or a nonlinear analysis and taking into account or ignoring shear deformation effect. From this figure, the influence of the shear deformation effect to the performed analysis is remarked. Moreover, in Table 4 the deflections and the bending moments at the ends $x = l$ and $x = 0$, respectively of the beam-column are presented for both of the aforementioned cases of analysis and taking into account or ignoring shear deformation effect. Finally, in Fig. 5 the deflection of the cantilever $w(l)$ versus the uniformly distributed load p_z is also shown for the aforementioned cases of analysis. From this figure, it is easily concluded that the nonlinear analysis and the shear deformation effect are of paramount importance.

Example 3 To demonstrate the range of applications of the proposed method, a free-free Timoshenko beam-column resting on a three-parameter foundation ($k_{Lz} = 35\,\text{MN/m}^2$, $k_{NLz} = \pm 3.5 \times 10^6\,\text{MN/m}^4$, $k_{Pz} =$

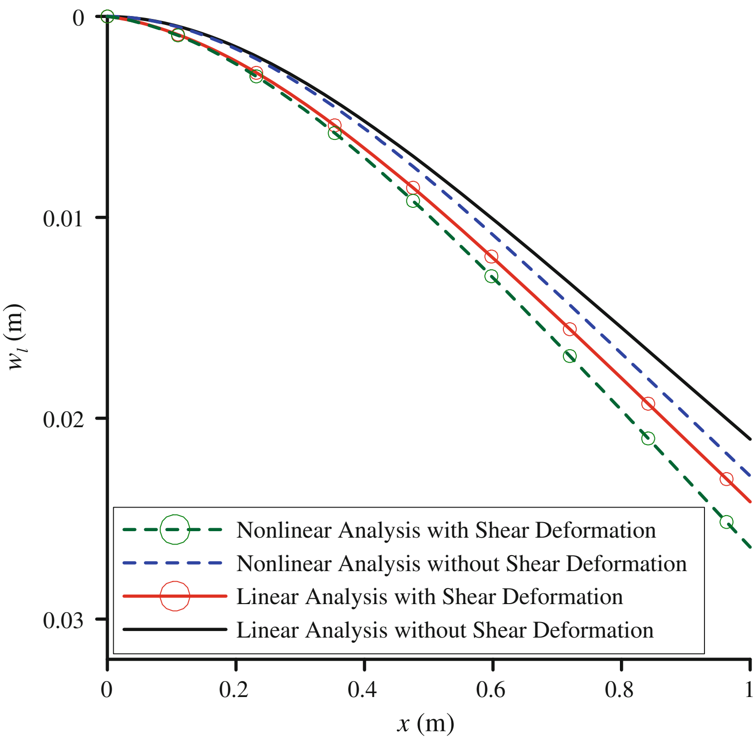

Fig. 4 Deflection w along the cantilever beam-column of Example 2

Table 4 Deflection (cm) and bending moment (kNm) at the ends $x = l$ and $x = 0$, respectively of the beam-column of Example 2

	Without shear deformation		With shear deformation	
	Linear analysis	Nonlinear analysis	Linear analysis	Nonlinear analysis
$w(l)$	2.10	2.29	2.42	2.64
$M_y(0)$	−94.53	−99.96	−96.64	−102.9

35 MN) is examined. The beam-column of length $l = 6.0$ m, ($E = 29 \times 10^6$ kN/m^2, $v = 0.2$, $A = 0.135$ m^2, $I_y = 1.013 \times 10^{-3}$ m^4, $a_z = 1.2$) is subjected to a concentrated axial force at its ends $P_x(0) = -P_x(l) = 600$ kN and to a concentrated transverse force at its midpoint $P_z(l/2) = 100$ kN.

In Figs. 6 and 7, the deflection w of the beam-column performing a nonlinear analysis is presented, for different types of foundation modeling taking into account or ignoring the tensionless character of the soil, respectively. Moreover, in Table 5 the deflections at $x = 0$ and the bending moments at $x = l/2$ of the beam-column are presented performing either a linear or a nonlinear analysis and taking into account or ignoring shear deformation effect. In Table 6, the extreme values of the displacement w and the foundation reaction p_{sz} of the Timoshenko beam-column performing a nonlinear analysis are presented, for different types of foundation modeling taking into account or ignoring the tensionless character of the soil. Finally, in Fig. 8 the displacement $w(l/2)$ versus the applied load $P_z(l/2)$ is presented for various types of foundation modeling illustrating the hardening and softening effect of the nonlinear foundation. The significant influence of the unilateral soil reaction to the deflections and the importance of the modeling of the subgrade to the response of the beam are verified.

Example 4 To demonstrate the importance of the unbonded contact between beam-column and subgrade, as the final example a pinned-pinned beam-column of length $l = 5$ m($E = 210$ GPa, $v = 0.3$, $A = 86.82 \times 10^{-4}\, m^2$, $I_y = 10.45 \times 10^{-5}$ m^4, $a_y = 1.462$, $a_z = 4.668$) resting on a three-parameter (either bilateral or unilateral) foundation reacting according to

$$p_{sz} = \gamma \cdot \left(U_w \left(5{,}000\,\mathrm{w} + 5{,}000\,\mathrm{w}^3 - 1{,}000 \frac{\partial^2 w}{\partial x^2} \right) \right) \tag{38}$$

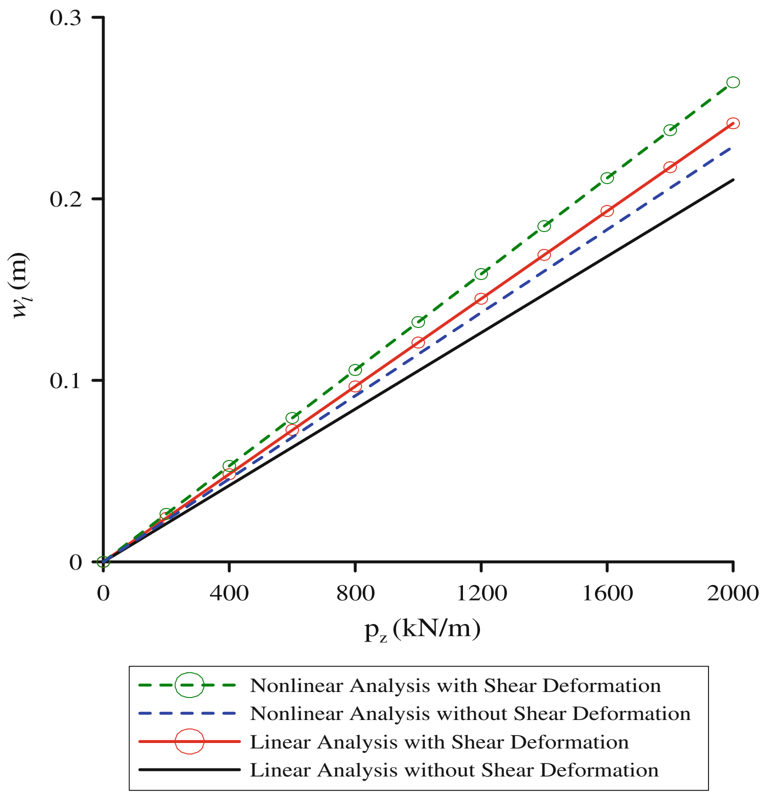

Fig. 5 Deflection $w(l)$ versus the uniformly distributed transverse load p_z, of the cantilever beam-column of Example 2

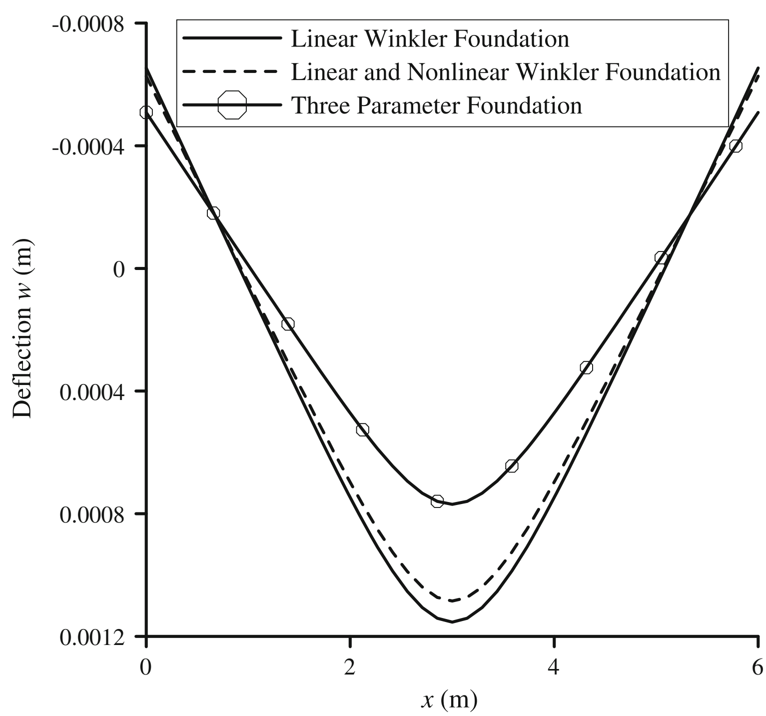

Fig. 6 Deflection w along the free beam-column of Example 3, for unilateral foundation

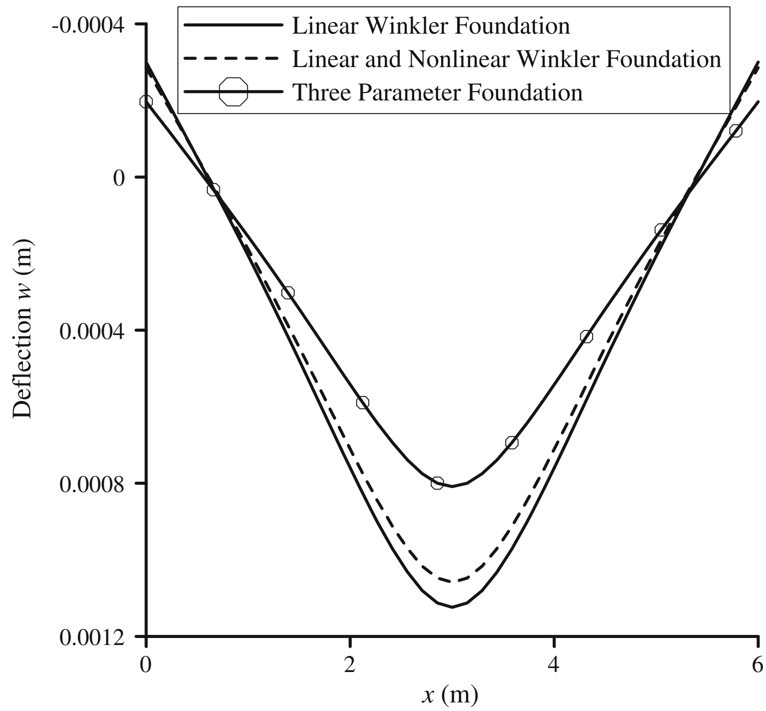

Fig. 7 Deflection w along the free beam-column of Example 3 for bilateral foundation

Table 5 Deflection $(\times 10^{-4}\,\text{m})$ at $x = 0$ and bending moment (kNm) at $x = l/2$ of the beam-column of Example 3, resting on a three-parameter foundation

	Without shear deformation		With shear deformation	
	Linear analysis	Nonlinear analysis	Linear analysis	Nonlinear analysis
$w(0)$	−4.60	−5.23	−4.46	−5.09
$M_y(l/2)$	23.0	23.5	23.8	24.2

Table 6 Extreme values of the deflections $(\times 10^{-4}\,\text{m})$ and the foundation reactions (kN/m) of the beam-column of Example 3

	Bilateral Winkler			Unilateral Winkler		
	Min w	Max w	Max p_{sz}	Min w	Max w	Max p_{sz}
Linear Winkler	−3.00	11.2	39.4	−6.53	11.5	40.4
Linear and nonlinear Winkler	−2.86	10.6	41.2	−6.27	10.8	42.4
Three parameter (positive nonlinearity)	−1.97	8.09	61.1	−5.09	7.69	61.6
Three parameter (negative nonlinearity)	−2.07	8.56	64.0	−5.23	8.06	63.9

is examined, where γ is a scale factor. The beam-column is subjected to a concentrated bending moment $M_y = 100\,\text{kNm}$ at its midpoint.

In Figs. 9 and 10, the deflection curves of the beam-column and the foundation reactions performing a nonlinear analysis and taking into account shear deformation effect are presented, respectively, for either bilateral or unilateral soil reaction and for two values of the factor γ. It is worth noting that the zero values of the soil reaction curve of Fig. 10 (solid lines) denote the detachment of the beam-column. Finally, in Table 7, the extreme values of the aforementioned quantities are given for both cases of bilateral and unilateral soil reaction and for various values of the factor γ. From the aforementioned figures and table, it is concluded that the unilateral character of the foundation is of paramount importance and cannot be ignored.

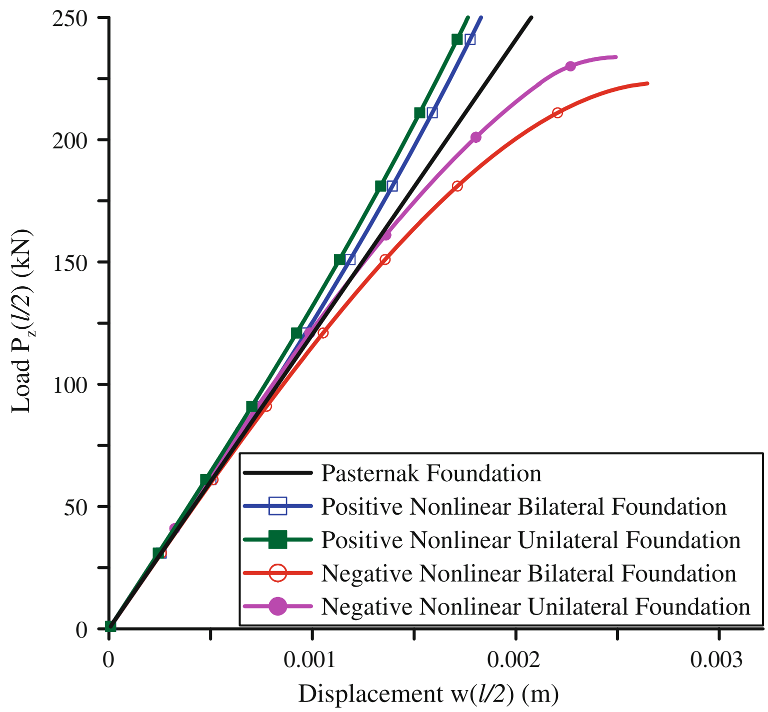

Fig. 8 Displacement $w(l/2)$ versus the applied load $P_z(l/2)$ of the free beam-column of Example 3

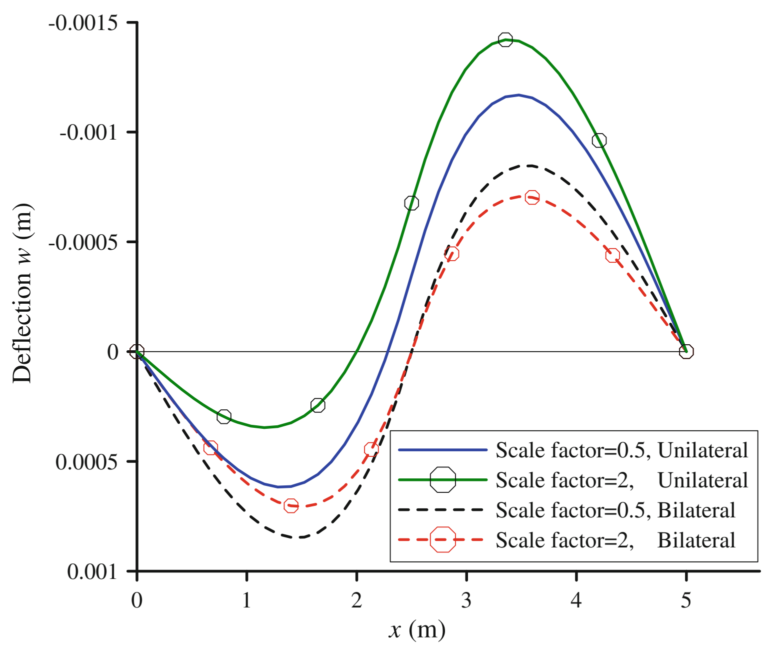

Fig. 9 Deflection curves of the beam-column of Example 4 resting on either a unilateral or a bilateral foundation

5 Concluding remarks

In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross-section, partially supported on tensionless three-parameter foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated

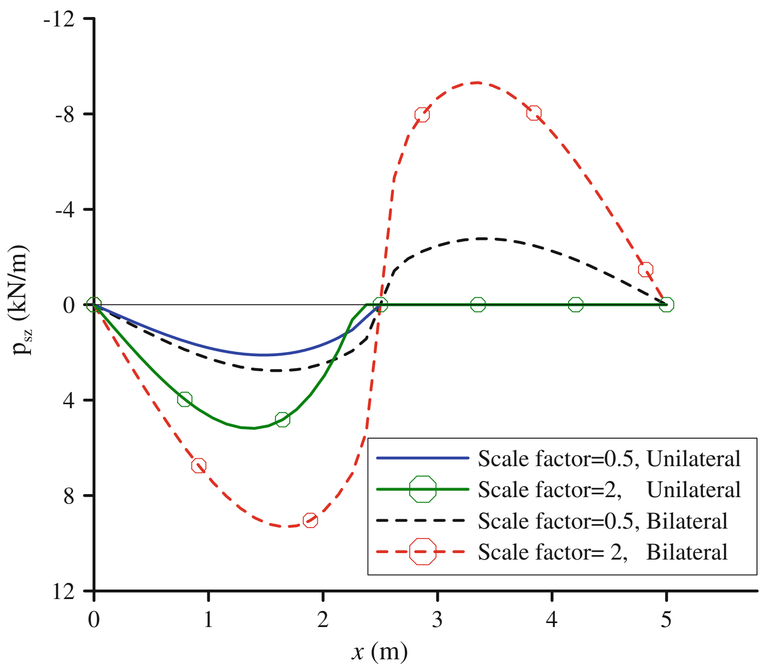

Fig. 10 Elastic foundation reactions of Example 4 for various values of the factor γ

Table 7 Extreme values of the displacements and the foundation reactions of the beam-column of Example 4

(γ)	Bilateral contact			Unilateral contact		
	Min w (mm)	Max w (mm)	Max p_{sz} (kN/m)	Min w (mm)	Max w (mm)	Max p_{sz} (kN/m)
0.5	−0.846	0.846	2.76	−1.17	0.617	2.11
1.0	−0.793	0.793	5.20	−1.30	0.475	3.38
1.5	−0.747	0.747	7.37	−1.38	0.394	4.33
2.0	−0.706	0.706	9.31	−1.42	0.346	5.18

transverse loading and bending moments in both directions as well as to axial loading. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. The main conclusions that can be drawn from this investigation are

(a) The numerical technique presented in this investigation is well suited for computer-aided analysis for beams of arbitrary simply or multiply connected doubly symmetric cross-section.
(b) In some cases, the effect of shear deformation is significant, especially for low beam slenderness values.
(c) The discrepancy of the obtained results performing a linear or a nonlinear analysis is remarkable.
(d) The significant influence of the unilateral character of the foundation in both the deflections and the soil reaction, especially in the case of a stiff soil is demonstrated.
(e) The inclusion of both the coupling effect of the linear elastic springs and the nonlinear character of the subgrade reaction influences the response of the beam and makes the modeling of the mechanical behavior of the subsoil more realistic and effective.
(f) The developed procedure retains most of the advantages of a BEM solution over a FEM approach, although it requires domain discretization.

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References

1. Kargarnovin, M.H., Younesian, D., Thompson, D.J., Jones, C.J.C.: Response of beams on nonlinear viscoelastic foundation to harmonic moving loads. Comput. Struct. **83**, 1865–1877 (2005)
2. Sharma, S., DasGupta, S.: The bending problem of axially constrained beams on elastic foundations. Int. J. Solids Struct. **11**, 853–859 (1975)
3. Kaschiev, M.S., Mikhajlov, K.: A beam resting on a tensionless Winkler foundation. Comput. Struct. **55**(2), 261–264 (1995)
4. Zhang, Y., Murphy, K.D.: Response of a finite beam in contact with a tensionless foundation under symmetric and asymmetric loading. Int. J. Solids Struct. **41**, 6745–6758 (2004)
5. Avramidis, I.E., Morfidis, K.: Bending of beams on three-parameter elastic foundation. Int. J. Solids Struct. **43**, 357–375 (2006)
6. Maheshwari, P.: Analysis of beams on tensionless reinforced granular fill-soil system. Int. J. Numer. Anal. Methods Geomech. **32**, 1479–1494 (2007)
7. Ma, X., Butterworth, J.W., Clifton, G.C.: Response of an infinite beam resting on a tensionless elastic foundation subjected to arbitrarily complex transverse loads. Mech. Res. Commun. **36**, 818–825 (2009)
8. Ma, X., Butterworth, J.W., Clifton, G.C.: Static analysis of an infinite beam resting on a tensionless Pasternak foundation. Eur. J. Mech. A Solids **28**, 697–703 (2009)
9. Zhang, Y.: Tensionless contact of a finite beam resting on Reissner foundation. Int. J. Mech. Sci. **50**, 1035–1041 (2008)
10. Ying, J., Lu, C.F., Chen, W.Q.: Two-dimensional elasticity solution for functionally graded beams resting on elastic foundations. Comput. Struct. **84**, 209–219 (2008)
11. Tullini, N., Tralli, A.: Static analysis of Timoshenko beam resting on elastic half-plane based on the coupling of locking-free finite elements and boundary integral. Comput. Mech. **45**, 211–225 (2010)
12. Silveira, R.A.M., Pereira, W.L.A, Goncalves, P.B.: Nonlinear analysis of structural elements under unilateral contact constraints by a Ritz type approach. Int. J. Solids Struct. **45**, 2629–2650 (2008)
13. Tsiatas, G.C.: Nonlinear analysis of non-uniform beams on nonlinear elastic foundation. Acta Mech. **209**, 141–152 (2010)
14. Katsikadelis, J.T.: The analog equation method. A boundary-only integral equation method for nonlinear static and dynamic problems in general bodies. Theor. Appl. Mech. **27**, 13–38 (2002)
15. Timoshenko, S.P., Goodier, J.N.: Theory of Elasticity. 3rd edn. McGraw-Hill, New York (1984)
16. Cowper, G.R.: The shear coefficient in Timoshenko's beam theory. J. Appl. Mech. ASME **33**(2), 335–340 (1966)
17. Schramm, U., Kitis, L., Kang, W., Pilkey, W.D.: On the shear deformation coefficient in beam theory. Finite Elem. Anal. Design **16**, 141–162 (1994)
18. Schramm, U., Rubenchik, V., Pilkey, W.D.: Beam stiffness matrix based on the elasticity equations. Int. J. Numer. Methods Eng. **40**, 211–232 (1997)
19. Stephen, N.G.: Timoshenko's shear coefficient from a beam subjected to gravity loading. ASME J. Appl. Mech. **47**, 121–127 (1980)
20. Hutchinson, J.R.: Shear coefficients for Timoshenko beam theory. ASME J. Appl. Mech. **68**, 87–92 (2001)
21. Ramm, E., Hofmann, T.J.: Stabtragwerke, Der Ingenieurbau. In: Mehlhorn, G. Band Baustatik/Baudynamik, Ernst&Sohn, Berlin (1995)
22. Rothert, H., Gensichen, V.: Nichtlineare Stabstatik. Springer, Berlin (1987)
23. Sapountzakis, E.J., Mokos, V.G.: A BEM solution to transverse shear loading of beams. Comput. Mech. **36**, 384–397 (2005)
24. Sapountzakis, E.J., Tsiatas, G.C.: Flexural-torsional buckling and vibration analysis of composite beams. Comput. Mater. Con. **6**(2), 103–115 (2007)
25. Sapountzakis, E.J., Mokos, V.G.: Analysis of plates stiffened by parallel beams. Int. J. Numer. Methods Eng. **70**, 1209–1240 (2007)
26. Powell, M.J.D.: On the quadratic programming algorithm of Goldfarb and Idnani. Math. Program. Study **25**, 46–61 (1985)
27. Powell, M.J.D.: Restart procedures for the conjugate gradient method. Math. Program. **12**, 241–254 (1977)