Nonlinear Analysis of Shear Deformable Beam-Columns Partially Supported on Tensionless Winkler Foundation

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Abstract. In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on tensionless Winkler foundation, undergoing moderate large deflections under general boundary conditions.

1. Introduction

In this paper, a boundary element method is developed for the nonlinear analysis of shear deformable beam-columns of arbitrary doubly symmetric simply or multiply connected constant cross section, partially supported on tensionless Winkler foundation, undergoing moderate large deflections under general boundary conditions. The beam-column is subjected to the combined action of arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to two stress functions and solved using the Analog Equation Method [1], a BEM based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as the shear forces along the span induced by the applied axial loading. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. Shear deformation effect is taken into account on the nonlinear analysis of beam-columns subjected to arbitrary loading (distributed or concentrated transverse loading and bending moments in both directions, as well as axial loading).
ii. The homogeneous linear half-space is approximated by a tensionless Winkler foundation.
iii. The beam-column is supported by the most general boundary conditions including elastic support or restrain, while its cross section is an arbitrary doubly symmetric one.
iv. The proposed model takes into account the coupling effects of bending and shear deformations along the member as well as shear forces along the span induced by the applied axial loading.
v. The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's and Cowper's definitions, for which several authors have pointed out that one obtains unsatisfactory results or definitions given by other researchers for which these factors take negative values.
vi. The effect of the material's Poisson ratio v is taken into account.
vii. The proposed method employs a BEM approach (requiring boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples are worked out to illustrate the efficiency, wherever possible the accuracy and the range of applications of the developed method.

2. Statement of the problem

Let us consider a prismatic beam-column of length l (Fig.1), of constant arbitrary doubly symmetric cross-section of area A. The homogeneous isotropic and linearly elastic material of the beam-column cross-section, with modulus of elasticity E, shear modulus G and Poisson's ratio v occupies the two dimensional multiply connected region Ω of the y,z plane and is bounded by the $\Gamma_j\left(j=1,2,...,K\right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig.1b Cyz is the principal bending coordinate system through the cross section's centroid. The beam-column is partially supported on a tensionless homogeneous elastic soil with k_x, k_y and k_z the moduli of subgrade reaction for the x, y, z directions, respectively (Winkler spring stiffness). Taking into account the unbonded contact between beam and subgrade, the interaction pressure at the interface is compressive and can be represented for the longitudinal and transverse directions by the following relations

$$p_{sx}=U_u\left(x\right)k_x u \qquad p_{sy}=U_v\left(x\right)k_y v \qquad p_{sz}=U_w\left(x\right)k_z v \tag{1a,b,c}$$

where $U_i\left(x\right)$ is the unit step function defined as

$$U_i\left(x\right)=\begin{cases}0 & if\ \ i>0\\ 1 & if\ \ i\leq 0\end{cases} \qquad i=u,v,w \tag{2}$$

The beam is subjected to the combined action of the arbitrarily distributed or concentrated axial loading $p_x=p_x\left(x\right)$, transverse loading $p_y=p_y\left(x\right)$, $p_z=p_z\left(x\right)$ acting in the y and z directions, respectively and bending moments $m_y=m_y\left(x\right)$, $m_z=m_z\left(x\right)$ along y and z axes, respectively (Fig.1a).

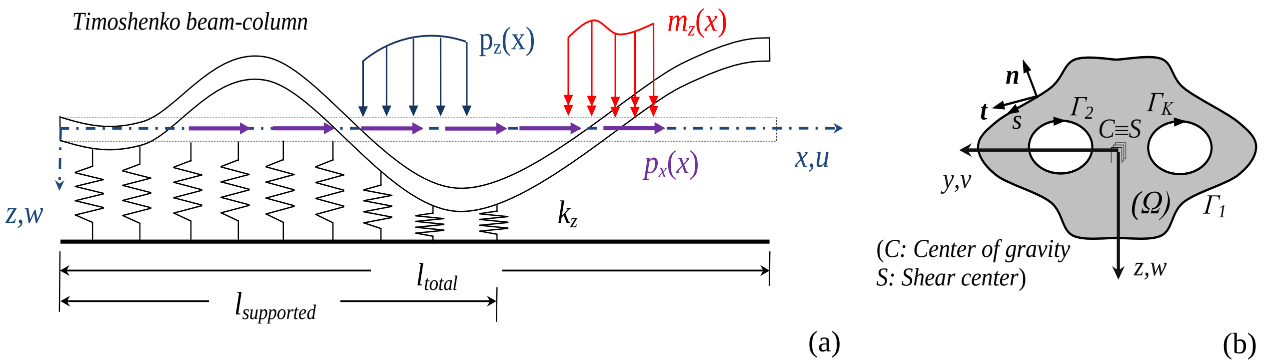

Fig. 1. x-z plane of a prismatic beam-column in axial - flexural loading (a) with an arbitrary doubly symmetric cross-section occupying the two dimensional region Ω (b).

Under the action of the aforementioned loading, the displacement field of the beam taking into account shear deformation effect is given as

$$\bar{u}\left(x,y,z\right)=u\left(x\right)-y\theta_z\left(x\right)+z\theta_y\left(x\right) \qquad \bar{v}\left(x\right)=v\left(x\right) \qquad \bar{w}\left(x\right)=w\left(x\right) \tag{3a,b,c}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u\left(x\right)$, $v\left(x\right)$, $w\left(x\right)$ are the corresponding components of the centroid C and $\theta_y\left(x\right)$, $\theta_z\left(x\right)$ are the angles of rotation due to bending of the cross-section with respect to its centroid.

Employing the strain-displacement relations of the three - dimensional elasticity for moderate displacements, the following strain components can be easily obtained

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \bar{v}}{\partial x}\right)^2 + \left(\frac{\partial \bar{w}}{\partial x}\right)^2\right] \tag{4a}$$

$$\gamma_{xz} = \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} + \left(\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial z}\right) \qquad \gamma_{xy} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} + \left(\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial y}\right) \tag{4b,c}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \tag{4d}$$

where it has been assumed that for moderate displacements $\left(\partial \bar{u}/\partial x\right)^2 << \partial \bar{u}/\partial x$, $\left(\partial \bar{u}/\partial x\right)\left(\partial \bar{u}/\partial z\right) << \left(\partial \bar{u}/\partial x\right) + \left(\partial \bar{u}/\partial z\right)$, $\left(\partial \bar{u}/\partial x\right)\left(\partial \bar{u}/\partial y\right) << \left(\partial \bar{u}/\partial x\right) + \left(\partial \bar{u}/\partial y\right)$. Substituting the displacement components (3) to the strain-displacement relations (4), the strain components can be written as

$$\varepsilon_{xx}(x,y,z) = u' + z\theta_y' - y\theta_z' + \frac{1}{2}\left(v'^2 + w'^2\right) \qquad \gamma_{xy} = v' - \theta_z \qquad \gamma_{xz} = w' + \theta_y \tag{5a,b,c}$$

where γ_{xy}, γ_{xz} are the additional angles of rotation of the cross-section due to shear deformation.

Considering strains to be small, employing the second Piola – Kirchhoff stress tensor and assuming an isotropic and homogeneous material, the stress components are defined in terms of the strain ones as

$$\begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \tag{6}$$

or employing eqns. (5) as

$$S_{xx} = E\left[u' + z\theta_y' - y\theta_z' + \frac{1}{2}\left(v'^2 + w'^2\right)\right] \qquad S_{xy} = G\cdot\left(v' - \theta_z\right) \qquad S_{xz} = G\cdot\left(w' + \theta_z\right) \tag{7a,b,c}$$

Considering a beam-column element of length dx at its deformed shape and equating the external loads with the internal reaction, the equations of equilibrium are written as

$$\frac{dN}{dx} - p_{sx} + p_x = 0 \tag{8}$$

$$\frac{dQ_y}{dx} - p_{sy} + p_y = 0 \qquad \frac{dQ_z}{dx} - p_{sz} + p_z = 0 \tag{9a,b}$$

$$\frac{dM_y}{dx} - Q_z + m_y = 0 \qquad \frac{dM_z}{dx} + Q_y + m_z = 0 \tag{9c,d}$$

where the stress resultants of the beam-column are given as

$$N = \int_\Omega S_{xx} d\Omega \qquad \mathrm{M}_y = \int_\Omega S_{xx} z d\Omega \qquad \mathrm{M}_z = -\int_\Omega S_{xx} y d\Omega \tag{10a,b,c}$$

$$Q_y = \int_{A_y} S_{xy} d\Omega \qquad Q_z = \int_{A_z} S_{xz} d\Omega \tag{10d,e}$$

Substituting the expressions of the stress components (7) into equations (10), the stress resultants are obtained as

$$N = EA\left[u' + \frac{1}{2}\left(v'^2 + w'^2\right)\right] \qquad M_y = EI_y\theta_y' \qquad M_z = EI_z\theta_z' \tag{11a,b,c}$$

$$Q_y = GA_y\gamma_{xy} \qquad Q_z = GA_z\gamma_{xz} \tag{11d,e}$$

where A is the cross section area, I_y, I_z the moments of inertia with respect to the principle bending axes given as

$$A = \int_\Omega d\Omega \tag{12}$$

$$I_y = \int_\Omega z^2 d\Omega \qquad I_z = \int_\Omega y^2 d\Omega \tag{13a,b}$$

and GA_y, GA_z are its shear rigidities of the Timoshenko's beam theory, where

$$A_y = \kappa_y A = \frac{1}{a_y} A \qquad A_z = \kappa_z A = \frac{1}{a_z} A \tag{14a,b}$$

are the shear areas with respect to y, z axes, respectively with κ_y, κ_z the shear correction factors and a_y, a_z the shear deformation coefficients. Substituting the stress resultants of eqns. (11) and the strain resultants of eqns. (5) in the equilibrium equations (8), (9) the differential equations of equilibrium are written as

$$-EA\left(u'' + w'w'' + v'v''\right) + U_u k_x u = p_x \tag{15a}$$

$$-\left(Nv'\right)' - GA_y\left(v'' - \theta_z'\right) + U_v k_y v = p_y \qquad -EI_z\theta_z'' - GA_y\left(v' - \theta_z\right) = m_z \tag{15b,c}$$

$$-\left(Nw'\right)' - GA_z\left(w'' + \theta_y'\right) + U_w k_z w = p_z \qquad -EI_y\theta_y'' + GA_z\left(w' + \theta_y\right) = m_y \tag{15d,e}$$

Combining equations (15b,c) and (15d,e), the governing differential equations with respect to u, v, w of a geometrically nonlinear Timoshenko beam-column, partially supported on a tensionless Winkler foundation, subjected to the combined action of axial and transverse loading are obtained as

$$-EA\left(u'' + w'w'' + v'v''\right) + U_u k_x u = p_x \tag{16a}$$

$$EI_z v'''' + \frac{EI_z}{GA_y}\left(Nv'\right)''' - \left(Nv'\right)' + \left(k_y v - \frac{EI_z}{GA_y}\left(k_y v''\right)\right)U_v = p_y - \frac{EI_z}{GA_y}\left(p_y''\right) - m_z' \tag{16b}$$

$$EI_y w'''' + \frac{EI_y}{GA_z}\left(Nw'\right)''' - \left(Nw'\right)' + \left(k_z w - \frac{EI_y}{GA_z}\left(k_z w''\right)\right)U_w = p_z - \frac{EI_y}{GA_z}\left(p_z''\right) + m_y' \tag{16c}$$

These equations are also subjected to the pertinent boundary conditions, which are given as

$$a_1 u(x) + \alpha_2 N(x) = \alpha_3 \tag{17}$$

$$\beta_1 v(x) + \beta_2 V_y(x) = \beta_3 \qquad \bar{\beta}_1\theta_z(x) + \bar{\beta}_2 \mathrm{M}_z(x) = \bar{\beta}_3 \tag{18a,b}$$

$$\gamma_1 w(x) + \gamma_2 V_z(x) = \gamma_3 \qquad \bar{\gamma}_1\theta_y(x) + \bar{\gamma}_2 \mathrm{M}_y(x) = \bar{\gamma}_3 \tag{19a,b}$$

at the beam ends $x = 0, l$. In eqns. (18), (19) V_y, V_z and M_y, M_z are the reactions and bending moments with respect to y, z axes, respectively, which together with the angles of rotation due to bending θ_y, θ_z are given as

$$V_y = -EI_z v''' - \frac{EI_z}{GA_y}\left[Nv''' - U_v k_y v'\right] + Nv' \qquad V_z = -EI_y w''' - \frac{EI_y}{GA_z}\left[Nw''' - U_w k_z w'\right] + Nw' \tag{20a,b}$$

$$M_z = EI_z v'' + \frac{EI_z}{GA_y}\left[Nv'' - U_v k_y v\right] \qquad M_y = -EI_y w'' - \frac{EI_y}{GA_z}\left[Nw'' - U_w k_z w\right] \tag{20c,d}$$

$$\theta_y = \frac{EI_y}{G^2 A_z^{\,2}}\left(U_w k_z w' - \left(Nw'\right)''\right) - \frac{1}{GA_z}\left(EI_y w''' + GA_z w'\right) \tag{20e}$$

$$\theta_z = \frac{EI_z}{G^2 A_y^{\,2}}\left(\left(Nv'\right)'' - U_v k_y v'\right) + \frac{1}{GA_y}\left(EI_z v''' + GA_y v'\right) \tag{20f}$$

Finally, $\alpha_j, \beta_j, \bar{\beta}_j, \gamma_j, \bar{\gamma}_j$ ($j = 1,2,3$) are functions specified at the beam-column ends $x = 0, l$. Eqns. (17)-(19) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restraint. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1 = \beta_1 = \gamma_1 = 1$, $\bar{\beta}_1 = \bar{\gamma}_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\gamma}_2 = \bar{\gamma}_3 = 0$).

The solution of the boundary value problem given from eqns. (16), subjected to the boundary conditions (17)-(19), which represents the nonlinear flexural analysis of a Timoshenko beam-column, partially supported on a tensionless Winkler foundation, presumes the evaluation of the shear deformation coefficients a_y, a_z, corresponding to the principal coordinate system Cyz. These coefficients are established equating the approximate formula of the shear strain energy per unit length

$$U_{appr.} = \frac{a_y Q_y^2}{2AG} + \frac{a_z Q_z^2}{2AG} \tag{21}$$

with the exact one given from

$$U_{exact} = \int_\Omega \frac{\left(\tau_{xz}\right)^2 + \left(\tau_{xy}\right)^2}{2G} d\Omega \tag{22}$$

and are obtained as [2]

$$a_y = \frac{1}{\kappa_y} = \frac{A}{\Delta^2}\int_\Omega \left[(\nabla\Theta) - \boldsymbol{e}\right]\cdot\left[(\nabla\Theta) - \boldsymbol{e}\right] d\Omega \qquad a_z = \frac{1}{\kappa_z} = \frac{A}{\Delta^2}\int_\Omega \left[(\nabla\Phi) - \boldsymbol{d}\right]\cdot\left[(\nabla\Phi) - \boldsymbol{d}\right] d\Omega \tag{23a,b}$$

where $\left(\tau_{xz}\right)_j, \left(\tau_{xy}\right)_j$ are the transverse (direct) shear stress components, $(\nabla) \equiv \boldsymbol{i_y}(\partial/\partial y) + \boldsymbol{i_z}(\partial/\partial z)$ is a symbolic vector with $\boldsymbol{i_y}, \boldsymbol{i_z}$ the unit vectors along y and z axes, respectively, Δ is given from

$$\Delta = 2(1+\nu) I_y I_z \tag{24}$$

ν is the Poisson ratio of the cross section material, $\boldsymbol{e}$ and $\boldsymbol{d}$ are vectors defined as

$$\boldsymbol{e} = \left(\nu I_y \frac{y^2 - z^2}{2}\right)\boldsymbol{i_y} + \nu I_y yz \boldsymbol{i_z} \qquad \boldsymbol{d} = \nu I_z yz \boldsymbol{i_y} - \left(\nu I_z \frac{y^2 - z^2}{2}\right)\boldsymbol{i_z} \tag{25a,b}$$

and $\Theta(y,z)$, $\Phi(y,z)$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems [2]

$$\nabla^2\Theta = -2I_y y \quad \text{in} \quad \Omega \qquad \frac{\partial\Theta}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{e} \quad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{26a,b}$$

$$\nabla^2\Phi = -2I_z z \quad \text{in} \quad \Omega \qquad \frac{\partial\Phi}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{d} \quad \text{on} \quad \Gamma = \bigcup_{j=1}^{K+1}\Gamma_j \tag{27a,b}$$

where $\boldsymbol{n}$ is the outward normal vector to the boundary Γ. In the case of negligible shear deformations $a_y = a_z = 0$. It is also worth here noting that the boundary conditions (26b), (27b) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\boldsymbol{n}$ vanishes on the free surface of the beam.

3. Integral Representations – Numerical Solution

According to the precedent analysis, the nonlinear flexural analysis of a Timoshenko beam-column, partially supported on a tensionless Winkler foundation, undergoing moderate large deflections reduces in establishing the displacement components $u(x)$ and $v(x)$, $w(x)$ having continuous derivatives up to the second and up to the fourth order with respect to x, respectively. Moreover, these displacement components must satisfy the coupled governing differential equations (16) inside the beam and the boundary conditions (17)-(19) at the beam ends $x = 0, l$. Eqns. (16) are solved using the Analog Equation Method [1] as it is developed for hyperbolic differential equations [3].

4. Numerical examples

In order to illustrate the importance of the nonlinear analysis and the influence of the shear deformation effect, a clamped beam-column of length $l = 5m$, having a hollow rectangular cross section ($E = 210\,GPa$, $v = 0.3$) and resting on a homogeneous (either bilateral or unilateral) elastic foundation of stiffness k_z, as this is shown in Fig.2, is examined.

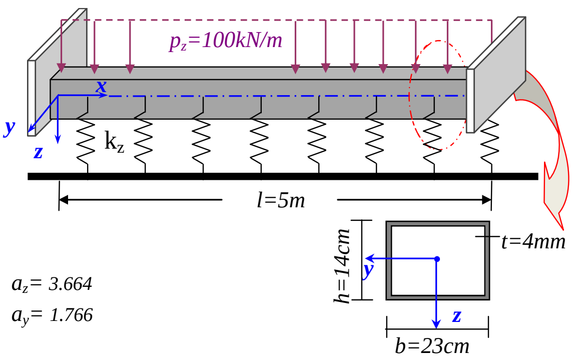

Fig. 2. Clamped beam-column of hollow rectangular cross section subjected to the uniformly distributed load (case i).

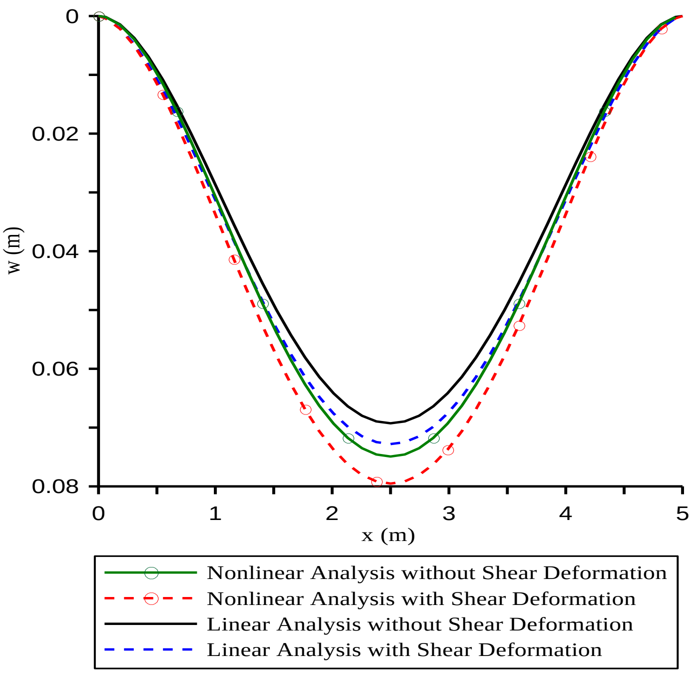

Fig. 3. Deflection w along the beam-column (case i) for soil stiffness $k_z = 50kN/m^2$.

In Fig.3 the deflection w along the beam-column resting on a tensionless foundation with $k_z = 50kN/m^2$ and subjected to a uniformly distributed load $p_z = 100kN/m$ (case i) are presented performing either a linear or a nonlinear analysis and taking into account or ignoring shear deformation effect. From this figure, the influence of the nonlinearity to the performed analysis is remarked, while the discrepancy of the obtained results due to the shear deformation effect justifies its inclusion even in thin walled sections. Moreover, in Table 1 the deflection and the bending moment at the midpoint of the beam-column are presented performing either a linear or a nonlinear analysis and taking into account or ignoring shear deformation effect. Finally, in Fig.4 the deflection curves of the beam-column resting on a tensionless foundation are presented for various values of the modulus k_z of the subgrade reaction, performing a nonlinear analysis, taking into account shear deformation effect and demonstrating the importance of the soil stiffness in the obtained results.

To illustrate the importance of the tensionless character of the subgrade reaction, the same beam-column subjected to a concentrated moment $M_y = -100kNm$ at its midpoint (case ii) is also studied. In Fig.5 the deflection curves of the beam-column resting on a tensionless foundation are presented for various values of the modulus k_z of the subgrade reaction, performing a nonlinear analysis and taking into account shear deformation effect. Additionally, in Table 2 the extreme values of the displacements and the soil reaction are presented for both cases of bilateral and unilateral soil reaction for various values of the modulus k_z performing a geometrical nonlinear analysis and taking into account shear deformation effect. From the aforementioned figure and table, it is concluded that the unilateral character of the foundation is of paramount importance and the error occurred from the ignorance of this behavior is considerable.

	Without Shear Deformation		With Shear Deformation	
	Linear Analysis	Nonlinear Analysis	Linear Analysis	Nonlinear Analysis
$w_{(l/2)}$	7.49	6.93	7.95	7.28
$M_{y(l/2)}$	188.23	178.13	189.57	176.89

Table 1. Deflection (cm) and moment (kNm) at the midpoint of the clamped beam-column (case i), for $k_z = 50kN/m^2$.

k_z (kN/m^2)	Bilateral Winkler			Unilateral Winkler		
	Min w (mm)	Max w (mm)	Max p_{sz} (kN/m)	Min w (mm)	Max w (mm)	Max p_{sz} (kN/m)
$5 \cdot 10^1$	-5.59	5.59	0.279	-5.67	5.53	0.276
$5 \cdot 10^2$	-5.39	5.39	2.694	-6.10	4.92	2.459
$5 \cdot 10^3$	-4.01	4.01	20.007	-7.84	2.56	12.822
$5 \cdot 10^4$	-1.45	1.45	72.675	-9.21	0.57	28.327

Table 2. Extreme values of the displacements and the foundation reaction of the beam-column (case ii).

5. Concluding remarks

The main conclusions that can be drawn from this investigation are

a. The numerical technique presented in this investigation is well suited for computer aided analysis for beams of arbitrary simply or multiply connected doubly symmetric cross section.

b. In some cases, the effect of shear deformation is significant, especially for low beam slenderness values.
c. The discrepancy of the obtained results performing a linear or a nonlinear analysis is remarkable.
d. The significant influence of the unilateral character of the foundation in both the deflections and the soil reaction, especially in the case of a stiff soil is demonstrated.
e. The importance of the soil stiffness to the response of the beam – column is verified.
f. The developed procedure retains most of the advantages of a BEM solution over a FEM approach, although it requires domain discretization.

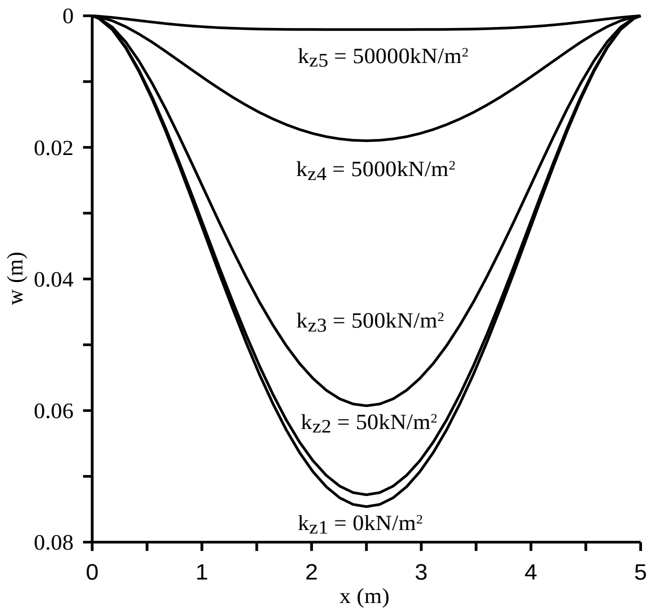

Fig. 4. Deflection curves of the beam-column (case i) for various values of the modulus k_z of the subgrade reaction.

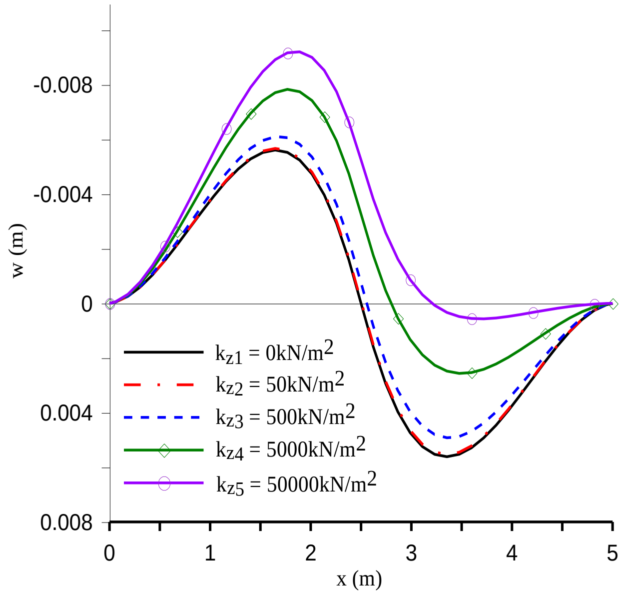

Fig. 5. Deflection curves of the beam-column (case ii) for various values of the modulus k_z of the tensionless subgrade reaction.

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