Abstract

In this investigation the inelastic analysis of beams of doubly symmetric simply or multiply connected constant cross section resting on inelastic foundation is presented employing the boundary element method. The beam is subjected to arbitrarily distributed or concentrated bending loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modelled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative solution strategy is adopted to restore global equilibrium along with an efficient iterative process to integrate the inelastic rate equations. The arising boundary value problem is solved employing the boundary element method. Numerical results are worked out to illustrate the method, demonstrate its efficiency and wherever possible its accuracy.

Keywords: beam on nonlinear foundation, Winkler foundation, inelastic analysis, distributed plasticity, boundary element method

1 Introduction

In engineering practice we often come across the analysis of beams on or in soil foundation. Piles, pile-columns and pile groups embedded in the soil medium as well as beams-columns resting on the soil half space are the most common examples. Analyses of civil engineering structures based on elastic constitutive equations are most likely to lead to extremely conservative designs not only due to significant difference between first yield in a cross section and full plasticity but also due to the unaccounted for yet significant reserves of strength that are enabled only after inelastic redistribution along members takes place. Thus, material nonlinearity is important for investigating the ultimate strength of a beam that resists bending loading, while distributed plasticity models are acknowledged in the literature [1-3] to capture more rigorously material nonlinearities than cross sectional stress resultant approaches [4] or lumped plasticity idealizations [5, 6].

According to the modeling of the mechanical behaviour of the soil-foundation interaction, the earliest model adopted is the Winkler elastic foundation [7]. In this model the supporting soil behaviour is approximated by a series of closely spaced, mutually independent, linear elastic vertical spring elements, providing resistance in direct proportion to the deflection of the beam.

The inelastic analysis of beams resting on soil medium are mainly limited in elastic beams on tensionless foundation. Kaschiev and Mikhajlov [8] presented a finite element solution for beams subjected to arbitrary loading. Later Zhang and Murphy [9] presented for the same problem an analytical/numerical solution making no assumption about either the contact area or the kinematics associated with the transverse deflection of the beam. Maheshwari [10] employed the finite difference method with the help of appropriate boundary and continuity conditions for the analysis of beams on tensionless reinforced granular fill-soil system, while Ma et. al. [11] used the transfer displacement function method (TDFM) to present the response of an infinite beam resting on a tensionless elastic foundation subjected to arbitrarily complex transverse loads. Finally, Sapountzakis and Kampitsis [12] employing the boundary element method presented the nonlinear analysis of shear deformable beam-columns partially supported on tensionless Winkler foundation, undergoing moderate large deflections.

To take into account the nonlinear nature of the soil Beaufait and Hoadley [13] used the midpoint difference method to solve the linear elastic problem of a beam supported on elastic foundation subjected to axial and distributed loading and an iterative approach to solve the nonlinear problem. In this model the constitutive law of the foundation follows a nonlinear p-y relation, while the beam is assumed elastic.

Models in which both the soil and the beam are considered to follow inelastic behaviour are considered very complex due to the combined effects of beam and springs plastification. Ayoub [14] proposed a new finite element formulation based on the mixed approach which is capable of capturing the inelastic behavior of both the beam and the foundation while Mullapudi and Ayoub [15] presented an inelastic element for the analysis of beams resting on two-parameter foundations. The element is derived from a two field mixed formulation with independent approximation of forces and displacements. The values for the two parameters of the foundation are derived through an iterative technique that is based on an assumption of plane strain for the soil medium.

In this investigation the inelastic analysis of beams of doubly symmetric simply or multiply connected constant cross section resting on inelastic foundation is presented employing the boundary element method. The beam is subjected to arbitrarily distributed or concentrated bending loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modelled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative solution strategy is adopted to restore global equilibrium along with an efficient iterative process to integrate the inelastic rate equations [16]. The arising boundary value problem is solved employing the boundary element method [17]. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. The formulation is a displacement based one taking into account inelastic redistribution along the beam axis by exploiting material constitutive laws and numerical integration over the cross sections (distributed plasticity approach).
ii. The inelasticity of the soil medium is taken into account.
iii. An incremental - iterative solution strategy is adopted to restore global equilibrium of the beam. Integration of the inelastic rate equations is performed for each monitoring station with an efficient iterative process and stress resultants are obtained employing incremental strains.
iv. The beam is supported by the most general nonlinear boundary conditions including elastic support or restrain, while its cross section is an arbitrarily doubly symmetric one.
v. To the authors' knowledge, the BEM has not yet been used for the solution of the aforementioned problem, while the developed procedure retains most of the advantages of a BEM solution over a pure domain discretization method.

Numerical results are worked out to illustrate the method, demonstrate its efficiency and wherever possible its accuracy.

2 Statement of the problem

2.1 Displacements, strains, stresses

Let us consider a prismatic beam of length l (Figure 1) with an arbitrarily shaped doubly symmetric constant cross section, occupying the two dimensional multiply connected region Ω of the y,z plane bounded by the $\Gamma_j \left(j=1,2,...,K \right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Figure 1b Cyz is the principal bending coordinate system through the cross section's centroid. The normal stress-strain relationship for the material is assumed to be elastic-plastic-strain hardening with initial modulus of elasticity and yield stress E and σ_{Y0}, respectively. The beam is supported on an inelastic soil which is characterized by the nonlinear Winkler modulus k. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading $p_z = p_z\left(x\right)$ and bending moment $m_y = m_y\left(x\right)$ acting in the x direction (Figure 1a).

Under the action of the aforementioned loading the displacement field of the beam is given as

$$\bar{u}(x,z) = u\left(x\right) + z\theta_y \tag{1a}$$

$$\bar{w}\left(x\right) = w\left(x\right) \tag{1b}$$

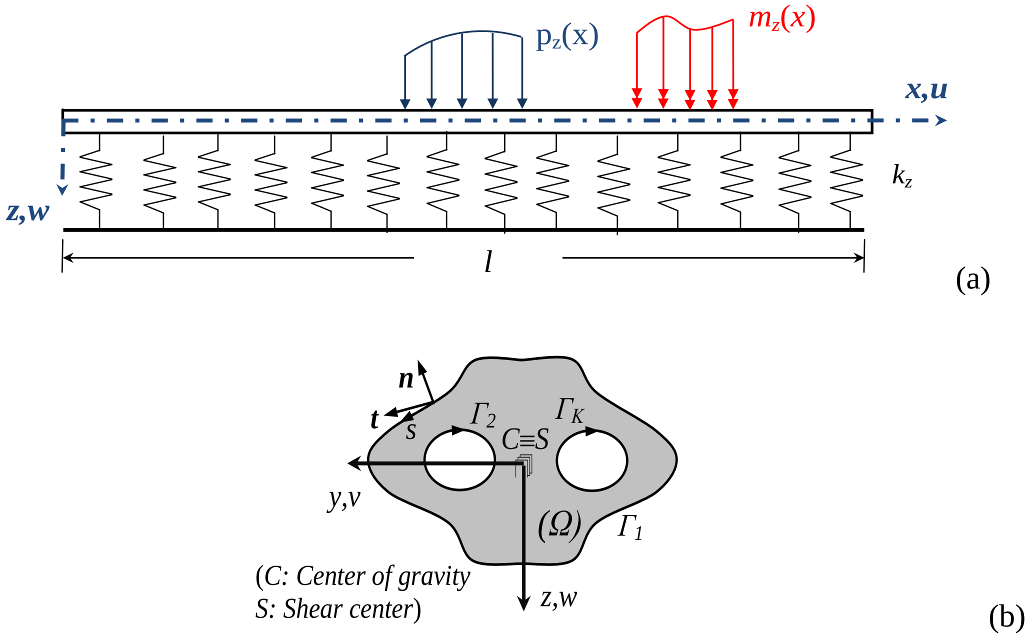

Figure 1. Prismatic beam resting on an inelastic foundation subjected to bending loading (a) with an arbitrary cross-section occupying the two dimensional region Ω (b)

where $\bar{u}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u(x)$, $w(x)$ are the corresponding components of the centroid C and $\theta_y(x,t)$ is the angle of rotation due to bending of the cross-section with respect to its centroid. Employing the strain-displacement relations considering small deflections and adopting the Euler-Bernoulli assumption the following strain components are obtained

$$\varepsilon_{xx} = -z\frac{d^2 w}{dx^2} \tag{2a}$$

$$\gamma_{xz} = 0 \Rightarrow \theta_y = -\frac{d^2 w}{dx^2} \tag{2b}$$

Considering strains to be small, employing the Cauchy stress tensor and assuming an isotropic and homogeneous material without exhibiting any damage during its plastification, the normal stress rate is defined in terms of the corresponding strain one as

$$d\sigma_{xx} = E^* d\varepsilon_{xx}^{el} \tag{3}$$

where $d(\cdot)$ denotes infinitesimal incremental quantities over time (rates), the superscript el denotes the elastic part of the strain component and $E^* = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$. If the plane stress hypothesis is undertaken then $E^* = \frac{E}{1-\nu^2}$ holds [18], while E is frequently considered instead of E^* ($E^* \approx E$) in beam formulations [18, 19]. This last consideration has been followed throughout the paper, while any other reasonable expression of E^* could also be used without any difficulty in many beam formulations.

As long as the material remains elastic or elastic unloading occurs ($d\varepsilon_{xx} = d\varepsilon_{xx}^{el}$) the stress rate is given with respect to the strain one from eqn. (3). If plastic flow occurs then $d\varepsilon_{xx} = d\varepsilon_{xx}^{el} + d\varepsilon_{xx}^{pl}$, where the superscript pl denotes the plastic part of the strain component. The yielding criterion is considered ignoring the interaction of normal and shear stresses and the yield condition is satisfied when the normal stress is equated with the yield stress of the material, that is

$$\sigma_{xx} - \sigma_Y\left(\varepsilon_{eq}^{pl}\right) = 0 \tag{4}$$

where σ_Y is the yield stress of the material and ε_{eq}^{pl} is the equivalent plastic strain, the rate of which is defined in [20] and is given as $d\varepsilon_{eq}^{pl} = d\lambda$ ($d\lambda$ is the proportionality factor [20]). Moreover, the plastic modulus h is defined as $h = d\sigma_Y / d\varepsilon_{eq}^{pl}$ or $d\sigma_Y = h d\lambda$ and can be estimated from a tension test as $h = E_t E / (E - E_t)$ (Figure 2). Using the aforementioned relation linking the yield stress rate and the proportionality factor and exploiting the plastic loading condition, the stress rates - total strain rate relation is resolved as

$$d\sigma_{xx} = E_t d\varepsilon_{xx} = E_t\left(d\varepsilon_{xx}^{el} + d\varepsilon_{xx}^{pl}\right) \tag{5}$$

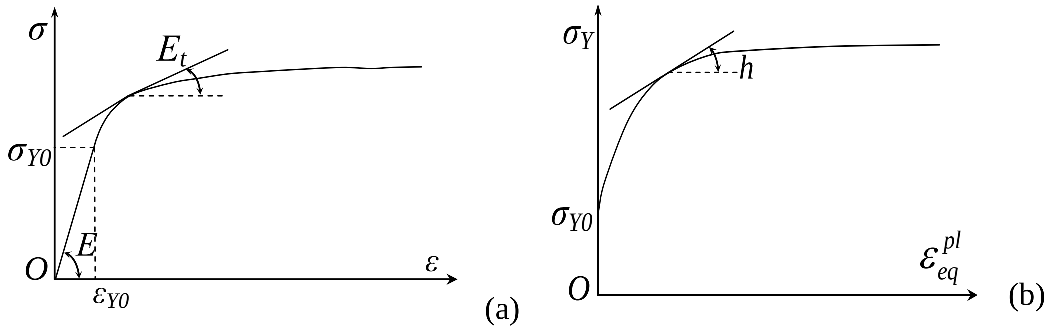

Figure 2. Normal stress - strain (a) and yield stress - equivalent plastic strain (b) relationships.

2.2 Equations of global equilibrium

To establish global equilibrium equations, the principle of virtual work neglecting body forces is employed, that is

$$\int_V \left(\sigma_{xx}\delta\varepsilon_{xx}\right)dV = \int_L \left(p_z\delta w - m_y\delta w'\right)dx - \int_L \left(kw\delta w\right)dx \tag{6}$$

where the integral quantities represent the strain energy and the external load work while $\delta(\cdot)$ denotes virtual quantities, and V is the volume of the beam.

In the elastic case, the well known fourth order governing differential equation of the beam on elastic foundation under bending loading is obtained

$$EI_y w'''' + kw = p_z + m_y' \tag{7}$$

where I_y is the moment of inertia with respect to the principle bending axis *y*, defined as

$$I_y = \int_\Omega z^2 d\Omega \tag{8}$$

In the inelastic case, eqn.(6) is reformulated as

$$\int_V \left(\sigma_{xx} + \Delta\sigma_{xx}\right)\delta\Delta\varepsilon_{xx}dV = \int_L \left(p_z\delta w - m_y\delta w'\right)dx - \int_L \left(k(w)w\delta w\right)dx \tag{6}$$

where $\Delta(\cdot)$ denotes incremental quantities (over loading steps). After some algebra through eqn (6), where the incremental strains are obtained employing eqns (2), the governing equation of the beam is obtained as

$$-\frac{d^2\Delta SM_y}{dx^2} = p_z(x) + \frac{dm_y(x)}{dx} - k(w)w + \frac{d^2 SM_y}{dx^2} \tag{7}$$

along with its corresponding boundary conditions

$$a_1\frac{d\Delta SM_y}{dx} + a_2\Delta w = a_3 - a_1\frac{dSM_y}{dx} \tag{8a}$$

$$\beta_1\Delta SM_y + \beta_2\frac{d\Delta w}{dx} = \beta_3 - \beta_1 SM_y \tag{8b}$$

where the stress resultant of the beam is defined as $SM_y = \int_\Omega \sigma_{xx} z d\Omega$ corresponding to the internal bending moment of the beam and the relevant incremental stress resultant is defined as $\Delta SM_y = \int_\Omega \Delta\sigma_{xx} z d\Omega$. Finally, α_i, β_i $(i = 1,2,3)$ are functions specified at the beam ends. The boundary conditions (8) are the most general ones for the problem at hand, including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) may be derived from eqns (8) by specifying appropriately the functions α_i and β_i (e.g. for a clamped edge it is $\alpha_2 = \beta_2 = 1$, $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 = 0$).

Eqns. (7)-(8) must be brought into a more convenient form in order to employ the boundary element technique for the numerical solution of the problem. The tangent modulus can be split into an elastic and a plastic part as

$$E_t = E - \frac{E^2}{E+h} \tag{9}$$

and the incremental stress resultant can be written as

$$\Delta SM_y = -EI_y \Delta w'' - I_1 \Delta w'' \tag{10}$$

In the above equations, I_1 is cross sectional property depending on the geometry of the cross section and the progress of its plastic region given as

$$I_1 = \int_\Omega \left(-\frac{E^2}{E+h} \right) z^2 d\Omega \tag{11}$$

Substituting eqn. (10) in eqns. (7), (8), the following boundary value problem is obtained

$$EI_y \Delta w'''' + I_1 \Delta w'''' + 2I_1' \Delta w''' + I_1'' \Delta w''' = p_z + \frac{dm_y}{dx} - kw + \frac{d^2 SM_y}{dx^2} \tag{12}$$

$$\alpha_1 \left(-EI_y \Delta w''' - I_1 \Delta w''' - I_1' \Delta w'' \right) + \alpha_2 \Delta w = \alpha_3 - \alpha_1 SM_y{}' \tag{13a}$$

$$\beta_1 \left(-EI_y \Delta w'' - I_1 \Delta w'' \right) + \beta_2 \Delta w' = \beta_3 - \beta_1 SM_y \tag{13b}$$

where I_1', I_1'' are the derivatives with respect to x of the cross sectional properties of eqn. (11).

3 Integral representations – numerical solution

3.1 Integral representations for the deflection w

According to the precedent analysis, the inelastic flexural analysis of Euler-Bernoulli beams resting on inelastic Winkler foundation reduces in establishing the displacement component $\Delta w(x)$ having continuous derivatives up to the fourth order with respect to x and satisfying the boundary value problem described by the governing differential equation (12) along the beam and the boundary conditions (13) at the beam ends $x = 0, l$.

The main difficulty of the problem at hand is the inhomogeneity of its governing equation. This boundary value problem (eqns (12), (13)) is solved employing the BEM [17], as this is developed in [21, 22] for the solution of a fourth order differential equation with constant coefficients, after some modifications. The motivation to use this particular technique is justified from the intention to retain the advantages of a BEM solution over a domain approach while keeping the use of shape functions to a minimum level.

3.2 Incremental - iterative solution algorithm

The modified or the fully nonlinear stiffness methods are usually employed for the incremental-iterative solution strategy of inelastic analysis of beams, that is the tangent stiffness matrix is kept constant throughout the whole increment or it is recomputed after each iteration, respectively. Apart from these methods, the initial stiffness method has also been implemented in the present study, since (i) it requires exclusively BEM computations to obtain the stiffness matrix (which is computed only once and stored in the beginning of the algorithm) and (ii) no convergence difficulties have arisen. Moreover, load control over the incremental steps is used and load stations are chosen according to the load history and convergence requirements. Usually, internal stress resultants and the externally applied loading are not in equilibrium, hence the out-of-balance forces do not vanish, thus an iterative process is initiated during an incremental step to restore equilibrium.

4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate wherever possible the accuracy, the efficiency and the range of applications of the developed method.

4.1 Example

For comparison purposes the elastic analysis of a free-free beam resting on Winkler elastic foundation subjected to a concentrated bending moment of $M_y = 50kNm$

acting at its midpoint, as shown in Figure 3 is examined. The elastic foundation is sandy clay with elastic modulus $E_s = 45.5MPa$, Poisson ration $v_s = 0.25$ and $\gamma = 1.0$. The beam and foundation parameters are presented in Table 1.

In Fig. 4 the deflection curve of the beam is presented as compared with the one obtained from the mixed formulation finite element method where the beam is discretized into four mixed elements with cubic moment interpolation functions [15]. Moreover, in Figs 5, 6 the bending moment distribution along the beam axis and the moment – rotation response of the midpoint are presented, respectively and compared with those obtained from [15]. From the presented results the accuracy of the proposed method is easily verified as the curves for all the aforementioned figures are almost identical.

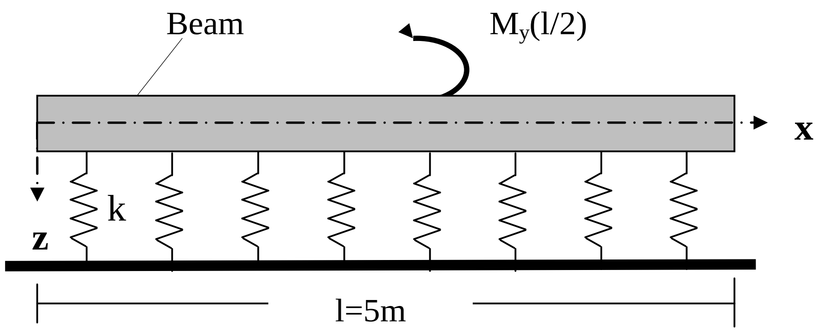

Figure 3. Prismatic elastic beam on elastic foundation subjected to a bending moment at its midpoint

$l\,(m)$	5	$E\,(MPa)$	10500
$I_y\,(m^4)$	0.4	$A\,(m^2)$	0.4
v	0.25	$k\,(MPa)$	3.081

Table 1. Geometric constants of the free-free beam and the Winkler foundation

As a variant of this example, the same free-free beam is analyzed taking into account the tensionless character of the foundation. The beam is subjected to a concentrate transverse force $P_z = 100kN$ and to a bending moment M_y at its midpoint. In Table 2 the rotation of the center of the beam for various values of the bending moment is presented as compared with those obtained from [15] showing the importance of the tensionless character of the soil. It is worth here noting that the data from the literature (Mullapudi and Ayoub [15]) were read from a graph and therefore are not overly accurate. Once again, the verification and the efficiency of the proposed method are illustrated.

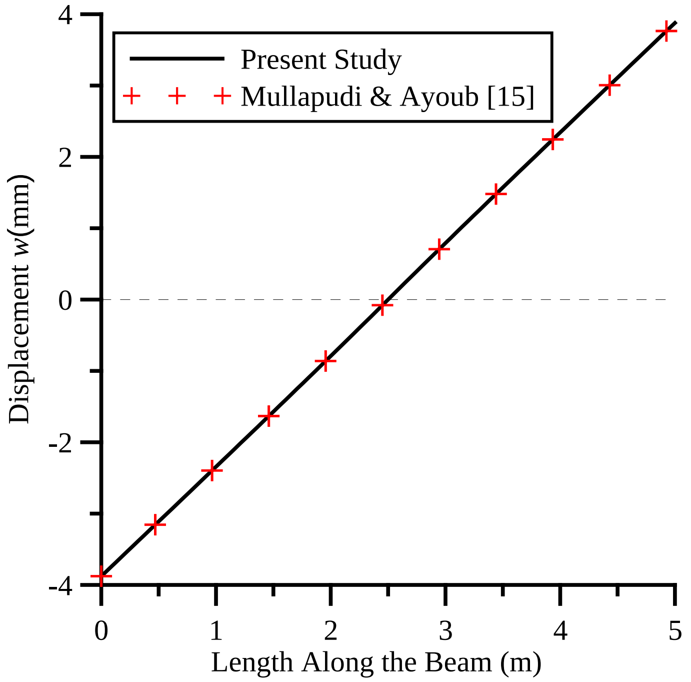

Figure 4. Deflection curve of the free-free beam on Winkler foundation

5 Concluding remarks

In this investigation the inelastic analysis of beams of doubly symmetric simply or multiply connected constant cross section resting on inelastic foundation is presented employing the boundary element method. The main conclusions that can be drawn from this investigation are

a) The numerical technique presented in this investigation is well suited for computer aided analysis of doubly symmetric simply or multiply connected constant cross section resting on inelastic foundation and subjected to the action of arbitrarily distributed or concentrated bending loading.

b) Accurate results are obtained using a relatively small number of nodal points across the longitudinal axis.

c) The developed procedure retains most of the advantages of a BEM solution over a domain approach.

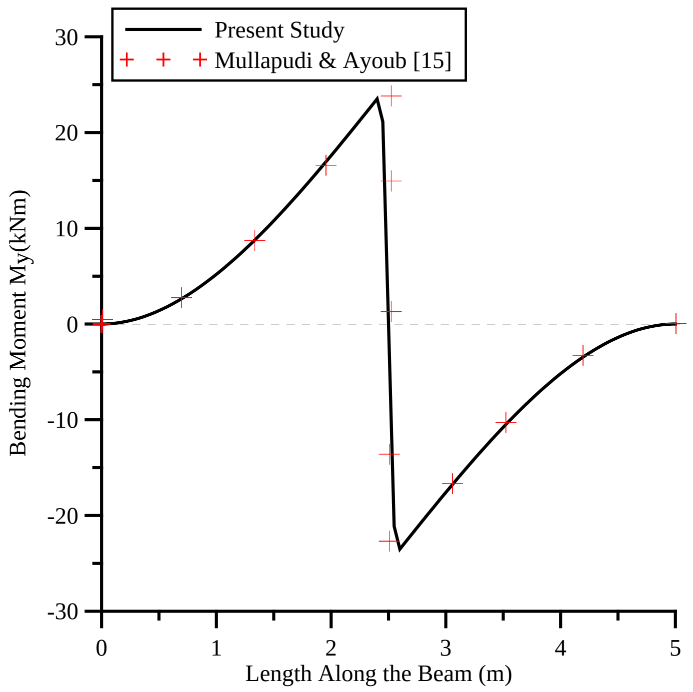

Figure 5. Bending moment distribution for the free-free beam on Winkler foundation

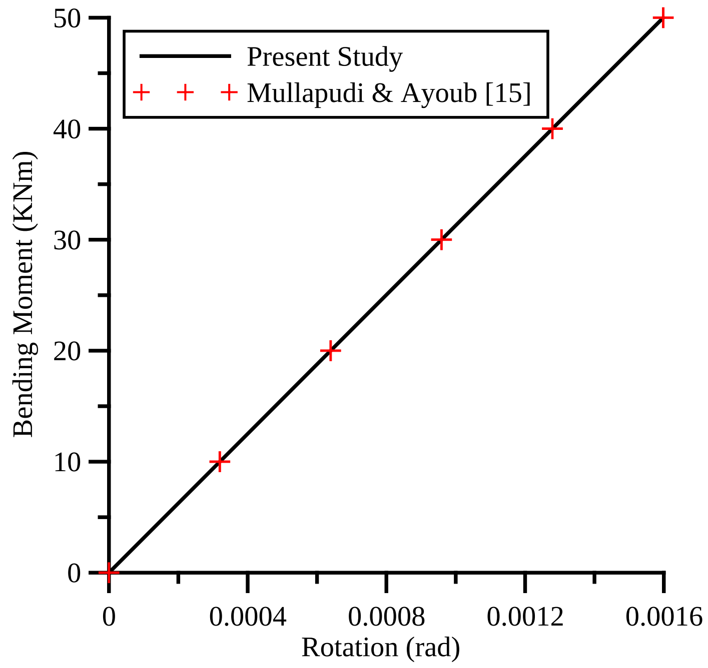

Figure 6. Moment – rotation response of the midpoint of the free-free beam on Winkler foundation

$M_y(l/2)(kNm)$	Present Analysis $\theta_y \times 10^{-3}(rad)$	Mullapudi and Ayoub [15] $\theta_y \times 10^{-3}(rad)$
25	0.798	0.790
50	1.596	1.594
75	2.395	2.393
100	3.300	3.239
125	4.769	4.636
150	7.462	7.400

Table 2. Rotation of the midpoint of the beam for various values of the bending moment

Acknowledgements

The work of this paper was conducted from the “DARE” project, financially supported by a European Research Council (ERC) Advanced Grant under the “Ideas” Programme in Support of Frontier Research [Grant Agreement 228254].

References

[1] P. Nukala, D. White, “A Mixed Finite Element for Three-Dimensional Nonlinear Analysis of Steel Frames”, Computer Methods in Applied Mechanics and Engineering 193, 2507-2545, 2004.

[2] L. Teh, M. Clarke, “Plastic-Zone Analysis of 3D Steel Frames Using Beam Elements”, Journal of Structural Engineering, 125, 1328-1337, 1999.

[3] A. Saritas, F.C. Filippou, “Frame Element for Metallic Shear-Yielding Members under Cyclic Loading”, Journal of Structural Engineering, 135, 1115-1123, 2009.

[4] M.R. Attalla, G.G. Deierlein, W. McGuire, “Spread of Plasticity: Quasi-Plastic-Hinge Approach”, Journal of Structural Engineering, 120, 2451-2473, 1994.

[5] J.G. Orbison, W. McGuire, J.F. Abel, “Yield Surface Applications in Nonlinear Steel Frame Analysis”, Computer Methods in Applied Mechanics and Engineering, 33, 557-573, 1982.

[6] C. Ngo-Huu, S. Kim, J. Oh, “Nonlinear Analysis of Space Steel Frames Using Fiber Plastic Hinge Concept”, Engineering Structures, 29, 649-657, 2007.

[7] M. Hetenyi, “Beams on Elastic Foundation”, The University of Michigan Press, Ann Arbor, 1946.

[8] S. Kaschiev, K. Mikhajlov, “A Beam Resting on a Tensionless Winkler Foundation”, Computers & Structures, 55(2), 261-264, 1995.
[9] K. Zhang, D. Murphy, “Response of a Finite Beam in Contact with a Tensionless Foundation under Symmetric and Asymmetric Loading”, International Journal of Solids and Structures, 41, 6745–6758, 2004.
[10] P. Maheshwari, “Analysis of Beams on Tensionless Reinforced Granular Fill-Soil System”, International Journal for Numerical and Analytical Methods in Geomechanics, 32, 1479-1494, 2007.
[11] X. Ma, J.W Butterworth, G.C. Clifton, “Response of an Infinite Beam Resting on a Tensionless Elastic Foundation Subjected To Arbitrarily Complex Transverse Loads”, Mechanics Research Communications, 36(7), 818-825, 2009.
[12] E.J. Sapountzakis, A.E. Kampitsis, “Nonlinear Analysis of Shear Deformable Beam-Columns Partially Supported on Tensionless Winkler Foundation”, International Journal of Engineering, Science and Technology, 2(4), 31-53, 2010.
[13] F.W Beaufait, P.W. Hoadley, “Analysis of Elastic Beams on Nonlinear Foundation”, Computers & Structures, 12, 669-676, 1980.
[14] A.Ayoub, “Mixed Formulation of Nonlinear Beam on Foundation Elements”, Computers & Structures, 81, 411-421, 2003.
[15] R. Mullapudi, A. Ayoub, “Nonlinear Finite Element Modelling of Beams on Two-Parameter Foundations”, Computer and Geotechnics, 37, 334-342, 2010.
[16] M. Ortiz, J. Simo, “Analysis of a New Class of Integration Algorithms for Elastoplastic Constitutive Relations”, International Journal for Numerical Methods in Engineering, 23, 353-366, 1986.
[17] J.T. Katsikadelis, “Boundary Elements: Theory and Applications”, Elsevier, Amsterdam-London, United Kingdom, 2002.
[18] V. Vlasov, “Thin-walled elastic beams”, Israel Program for Scientific Translations, Jerusalem, 1963.
[19] A.E. Armenakas, “Advanced Mechanics of Materials and Applied Elasticity” Taylor & Francis Group, New York, 2006.
[20] M.A. Crisfield, “Non-linear Finite Element Analysis of Solids and Structures”, Vol. 1: Essentials, John Wiley and Sons, New York, USA, 1991.
[21] E.J. Sapountzakis, “Solution of Non-Uniform Torsion of Bars by an Integral Equation Method”, Computers and Structures, **77**, 659-667, 2000.
[22] E.J. Sapountzakis, “Nonuniform Torsion of Multi-Material Composite Bars by the Boundary Element Method”, Computers and Structures, 79, 2805-2816, 2001.