Inelastic Analysis of Beams on Two Parameter Elastoplastic Foundation

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Abstract. In this investigation the inelastic analysis of beams of doubly symmetric simply or multiply connected constant cross section resting on two-parameter elastoplastic foundation is presented employing the boundary element method.

Introduction

In engineering practice we often come across the analysis of beams on/in soil medium. The beam-foundation design is commonly used in modeling piles, pile-columns and pile groups embedded in soil medium as well as beam-columns and railway tracks resting on soil half space. Moreover, design of beams and engineering structures based on elastic analysis are most likely to be extremely conservative not only due to significant difference between initial yield and full plastification in a cross section, but also due to the unaccounted for yet significant reserves of strength that are not mobilized in redundant members until after inelastic redistribution takes place.

In this paper, a boundary element method is developed for the inelastic analysis of beams of arbitrarily shaped doubly symmetric constant cross section resting on two-parameter tensionless elastoplastic foundation. The beam is subjected to arbitrarily distributed or concentrated vertical loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modelled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative solution strategy is adopted to resolve both the plastic part of stress resultants and the foundation reaction along with an efficient iterative process to integrate the inelastic rate equations [1]. The arising boundary value problem is solved employing the boundary element method [2]. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. The formulation is a displacement based one taking into account inelastic redistribution along the beam axis by exploiting material constitutive laws and numerical integration over the cross sections (distributed plasticity approach).
ii. The inelasticity of the soil medium is taken into account, employing two-parameter foundation model.
iii. The tensionless character of the foundation is also taken into consideration.
iv. An incremental-iterative solution strategy is adopted to restore global equilibrium of the beam.
v. The beam is supported by the most general nonlinear boundary conditions including elastic support or restrain, while its cross section is an arbitrarily doubly symmetric one.
vi. To the authors' knowledge, the BEM has not yet been used for the solution of the aforementioned problem, while the developed procedure retains most of the advantages of a BEM solution over a pure domain discretization method.

Numerical results are worked out to illustrate the method, demonstrate its efficiency and wherever possible its accuracy.

Statement of the problem

Let us consider a prismatic beam of length l (Fig. 1a) with an arbitrarily shaped doubly symmetric constant cross section, occupying the two dimensional multiply connected region Ω of the y,z plane bounded by the

$\Gamma_j\left(j=1,2,...,K\right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1b Cyz is the principal bending coordinate system through the cross section's centroid. The normal stress-strain relationship for the material is assumed to be elastic-plastic-strain hardening with initial modulus of elasticity and yield stress E and σ_{Y0}, respectively. The beam is resting on an inelastic two-parameter foundation which is characterized by the nonlinear force – deformation relation with Winkler and Pasternak initial stiffness k_w, k_p, yielding force P_{wY}, P_{pY} and hardening stiffness k_{wt}, k_{pt}, respectively. Thus, the foundation reaction assuming linear soil behavior is written as

$$p_s = k_w w - k_p \frac{d^2 w}{dx^2} \tag{1}$$

The beam is supported on an inelastic soil which is characterized by the nonlinear Winkler and Pasternak load-deformation laws. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading $p_z = p_z\left(x\right)$ and bending moment $m_y = m_y\left(x\right)$ acting in the x direction.

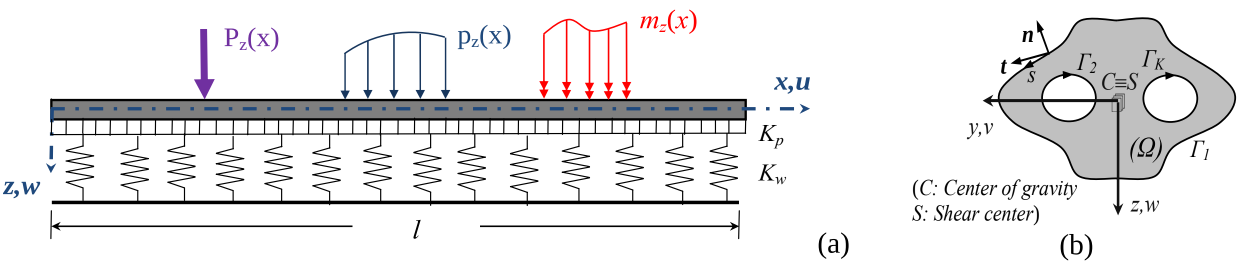

Fig. 1. Prismatic beam resting on an elastoplastic two-parameter foundation subjected to bending loading (a) with an arbitrary cross-section occupying the two dimensional region Ω (b)

Under the action of the aforementioned loading the displacement field of the beam is given as

$$\bar{u}(x,z) = u\left(x\right) + z\theta_y \qquad \bar{w}\left(x\right) = w\left(x\right) \tag{2a,b}$$

where $\bar{u}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u\left(x\right)$, $w\left(x\right)$ are the corresponding components of the centroid C and $\theta_y\left(x,t\right)$ is the angle of rotation due to bending of the cross-section with respect to its centroid. Employing the strain-displacement relations considering small deflections and adopting the Euler-Bernoulli assumption the following strain components are obtained

$$\varepsilon_{xx} = -z\frac{d^2 w}{dx^2} \qquad \gamma_{xz} = 0 \Rightarrow \theta_y = -\frac{d^2 w}{dx^2} \tag{3a,b}$$

Considering strains to be small, employing the Cauchy stress tensor and assuming an isotropic and homogeneous material without exhibiting any damage during its plastification, the normal stress rate is defined in terms of the corresponding strain one as

$$d\sigma_{xx} = E^* d\varepsilon_{xx}^{el} \tag{4}$$

where $d\left(\cdot\right)$ denotes infinitesimal incremental quantities over time (rates), the superscript el denotes the elastic part of the strain component and $E^* = \dfrac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)}$. If the plane stress hypothesis is undertaken then $E^* = \dfrac{E}{1-\nu^2}$ holds [3], while E is frequently considered instead of E^* ($E^* \approx E$) in beam formulations [3, 4]. This last consideration has been followed throughout the paper, while any other reasonable expression of E^* could also be used without any difficulty in many beam formulations.

As long as the material remains elastic or elastic unloading occurs ($d\varepsilon_{xx} = d\varepsilon_{xx}^{el}$) the stress rate is given with respect to the strain one from eq (4). If plastic flow occurs then $d\varepsilon_{xx} = d\varepsilon_{xx}^{el} + d\varepsilon_{xx}^{pl}$, where the superscript pl denotes the plastic part of the strain component. The yielding criterion is considered ignoring the interaction of normal and shear stresses and the yield condition is satisfied when the normal stress is equated with the yield stress of the material, that is

$$\sigma_{xx} - \sigma_Y \left(\varepsilon_{eq}^{pl} \right) = 0 \tag{5}$$

where σ_Y is the yield stress of the material and ε_{eq}^{pl} is the equivalent plastic strain, the rate of which is defined in [5] and is given as $d\varepsilon_{eq}^{pl} = d\lambda$ ($d\lambda$ is the proportionality factor [5]). Moreover, the plastic modulus h is defined as $h = d\sigma_Y \big/ d\varepsilon_{eq}^{pl}$ or $d\sigma_Y = hd\lambda$ and can be estimated from a tension test as $h = E_t E \big/ \left(E - E_t \right)$. Using the aforementioned relation linking the yield stress rate and the proportionality factor and exploiting the plastic loading condition, the stress rates - total strain rate relation is resolved as

$$d\sigma_{xx} = E_t d\varepsilon_{xx} = E_t \left(d\varepsilon_{xx}^{el} + d\varepsilon_{xx}^{pl} \right) \tag{6}$$

Equation of global equilibrium

To establish global equilibrium equations, the principle of virtual work neglecting body forces is employed, and the global equilibrium equations of the beam is obtained as

$$-\frac{d^2 SM_y}{dx^2} + k_w w - k_p \frac{d^2 w}{dx^2} = p_z\left(x\right) + \frac{dm_y\left(x\right)}{dx} \tag{7}$$

along with its corresponding boundary conditions

$$a_1 \frac{dSM_y}{dx} + a_2 w = a_3 \qquad \beta_1 SM_y + \beta_2 \frac{dw}{dx} = \beta_3 \tag{8a,b}$$

where the stress resultant corresponding to the internal bending moment of the beam is defined as

$$SM_y = \int_{\Omega} \sigma_{xx} z d\Omega \tag{9}$$

Since an incremental - iterative approach is adopted for the problem at hand, the incremental version of eq (7) is obtained as

$$-\frac{d^2 \Delta SM_y}{dx^2} + \Delta\left(k_w w\right) - \Delta\left(k_p \frac{d^2 w}{dx^2} \right) = \Delta p_z\left(x\right) + \frac{d\Delta m_y\left(x\right)}{dx} \tag{10}$$

where $\Delta(\cdot)$ denotes incremental quantities (over time), while the incremental stress resultants are given as

$$\Delta SM_y = -EI_y \Delta w'' - \Delta SM_y^{pl} \tag{11}$$

and ΔSM_y^{pl} is the plastic quantity defined as $\Delta SM_y^{pl} = \int_{\Omega} E\Delta\varepsilon^{pl} z d\Omega$. Similarly, the incremental version of the boundary conditions eqs (8) is also assumed.

Integral representation – Numerical solution

According to the precedent analysis, the inelastic flexural analysis of Euler-Bernoulli beams resting on elastoplastic Pasternak foundation reduces to establishing the displacement component $\Delta w\left(x\right)$ having continuous derivatives up to the fourth order with respect to x and satisfying the boundary value problem described by the governing differential equation (8) along the beam and the boundary conditions at the beam ends $x = 0, l$. This boundary value problem is solved employing the BEM [2], as this is developed in [6] for the solution of a fourth order differential equation with constant coefficients, after some modifications.

Incremental - iterative solution algorithm

The initial stiffness method has been implemented in the present study, since (i) it requires exclusively BEM computations to obtain the stiffness matrix (which is computed only once and stored in the beginning of the algorithm) and (ii) no convergence difficulties have arise. Moreover, load control over the incremental steps is used and load stations are chosen according to the load history and convergence requirements. Incremental stress resultants are decomposed into elastic and plastic part and they are computed through an iterative procedure, checking at each iteration if they have converged.

Numerical example

For comparison purposes, a simply supported beam of length $l = 300in$ and square cross section of side $d = 6.26in$ subjected to a monotonically increasing concentrated vertical load P at its midpoint has been studied. The beam's material is assumed to follow an elastic – plastic behavior with modulus of elasticity $E = 29000ksi$, yielding stress $\sigma_{Y0} = 30ksi$ and a strain hardening slope of 1.4% (tangent modulus $E_t = 406ksi$), while the Winkler foundation force – deformation relation is also considered to be elastic – plastic with initial stiffness equals to $k_w = 0.5kip / in^2$, yielding force $P_{wY} = 1.0k / in$ and hardening slope of 1.0% (tangent stiffness $k_{wt} = 5 \cdot 10^{-3} kip / in^2$). For the longitudinal discretization 20 linear elements have been employed, while the cross section has been subdivided into 36 quadrilateral cells (6 fibers) and a 2×2 Gauss integration scheme has been used for each cell.

The present example has been studied by Ayoub [7], developing both displacement and mixed-based finite element formulation capable of capturing the nonlinear behavior of both the beam and the foundation. The beam's section has been subdivided into 16 fibers, while for both the displacement and mixed models two different order of interpolation functions were used, employing a 6 element discretization. The results were compared with the converged solution obtained by a displacement-based model with fifth order polynomial and a mesh consisting of 32 elements.

In Fig. 2, the load - deformation curve at the beam's midpoint is presented, as compared with those obtained from FE formulation [7]. A very good agreement is once again verified between the corresponding curve of the present study and those of Ayoub. More specifically, the obtained curve is almost identical with the results of the mixed model with cubic moment function and the converged solution, which is assumed to describe the exact behavior of the beam - foundation system, while the other models present a perceptible amount of error in the inelastic region.

In Figs. 3, 4 the bending moment $M_y(x)$ and displacement $w(x)$ distribution along the beam's length are presented for the load stage producing vertical deformation at the midspan equal to 5, as compared with literature. The corresponding curves of the present study capture the exact behavior rather accurately and agree with the converged solution and the results of the mixed model with cubic moment interpolation function, while differ from the curves of the displacement model with cubic displacement function. Finally, in Table 1 the curvature at the midpoint $w''(l/2)$ and the foundation reaction p_s at $x = l/6$ are presented for the same load stage as compared with literature (the compared values have been extracted from a graph), verifying that the maximum values of the curvature and soil reaction are accurately represented. From these figures and table a very good agreement between the results of the proposed method and those of the literature is observed.

Concluding remarks

The main conclusions that can be drawn from this investigation are

a. The numerical technique presented in this investigation is well suited for computer aided analysis.
b. Accurate results are obtained using a relatively small number of nodal points across the longitudinal axis.
c. A small number of cells (fibers) is required in order to achieve satisfactory convergence.
d. The influence of both the inelastic and the tensionless character of the foundation is confirmed.
e. The developed procedure retains most of the advantages of a BEM solution over a domain approach.

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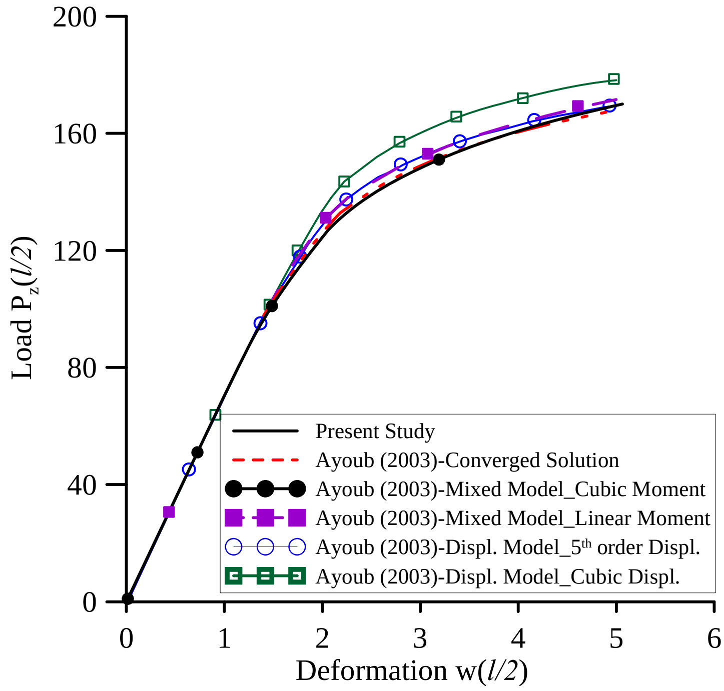

Fig. 2. Load – deformation curve at the midpoint of the beam.

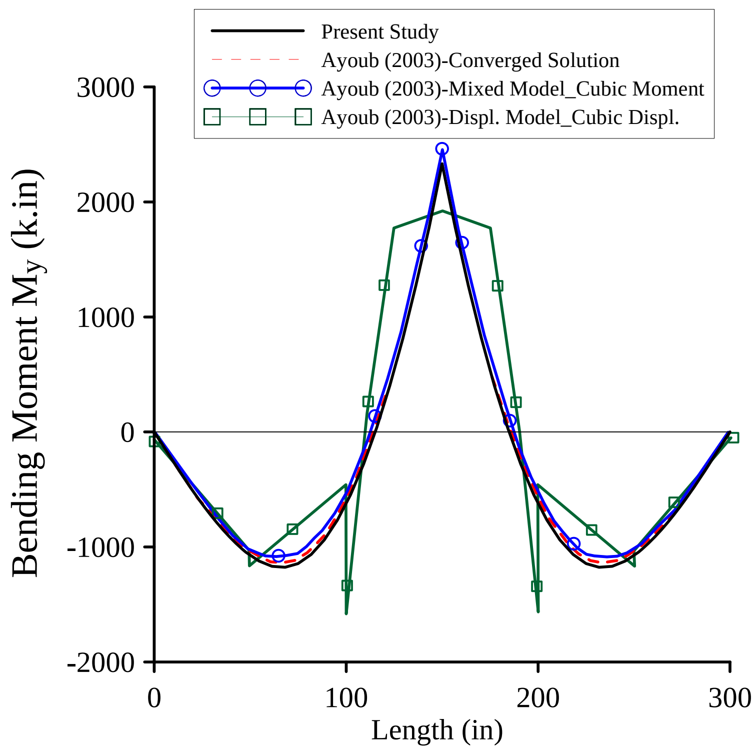

Fig. 3. Bending moment distribution $M_y(x)$ along the beam.

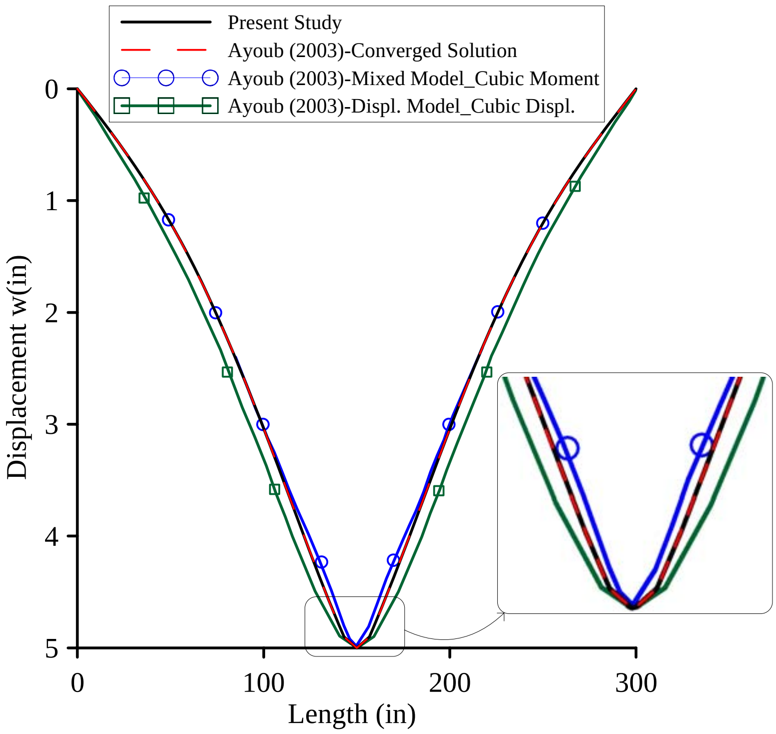

Fig. 4. Displacement distribution $w(x)$ *along the beam.*

	Present Analysis	Ayoub [7]				
		Converged Solution	Mixed Model		Displ. Model	
			cubic moment	linear moment	5^{th} order	cubic
$w''(l/2)\cdot 10^{-3}$ (1/in)	9.98	10.66	11.85	9.95	4.91	2.26
$p_s(l/6)$ (k/in)	0.601	0.60	0.60	0.611	0.619	0.689

Table 1. Curvature (1/in) and Foundation Reaction (k/in) of the beam.

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