Elastoplastic Dynamic Analysis of Beam-Foundation Systems Employing BEM

Andreas E. Kampitsis[1] and Evangelos J. Sapountzakis[2]

[1] School of Civil Engineering, National Technical University of Athens (NTUA)
Zografou Campus, GR-157 80, Athens, Greece. e-mail: cvakamb@gmail.com

[2] School of Civil Engineering, National Technical University of Athens (NTUA)
Zografou Campus, GR-157 80, Athens, Greece. e-mail: cvsapoun@central.ntua.gr

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Abstract. In this investigation a Boundary Element Method (BEM) is developed for the elastoplastic dynamic analysis of an Euler-Bernoulli beam of simply or multiply connected constant cross section having at least one axis of symmetry, resting on inelastic foundation.

Introduction

Beam-foundation systems which are subjected to dynamic loading often exhibit inelastic material behavior either concerning the structural's element or the foundation. Moreover, design of beams based on elastic analysis are most likely to be extremely conservative not only due to significant difference between initial yield and full plastification in a cross section, but also due to the unaccounted for yet significant reserves of strength that are not mobilized in redundant members until after inelastic redistribution takes place. Thus, material nonlinearity is important for investigating the ultimate strength of a beam that resists bending loading, while distributed plasticity models are acknowledged in the literature [1-3] to capture more rigorously material nonlinearities than cross sectional stress resultant approaches [4] or lumped plasticity idealizations [5,6]. Contrary to the good amount of attention in the literature concerning the elastic dynamic analysis of beams on elastic foundation, very little work has been done on the corresponding inelastic dynamic problems.

In this investigation a Boundary Element Method (BEM) is developed for the elastoplastic dynamic analysis of an Euler-Bernoulli beam of simply or multiply connected constant cross section having at least one axis of symmetry, resting on inelastic foundation. The beam is subjected to arbitrarily distributed or concentrated dynamic bending loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modeled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative time discretization scheme is adopted to restore global equilibrium along with an efficient iterative process to integrate the inelastic rate equations [7]. The arising boundary value problem is solved employing the boundary element method [8]. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. A BEM approach is employed for the dynamic elastoplastic analysis of a beam-foundation system.
ii. The formulation is a displacement based one taking into account inelastic redistribution along the beam axis by exploiting material constitutive laws and numerical integration over the cross sections (distributed plasticity approach).
iii. The inelasticity of the soil medium is taken into account.
iv. An incremental - iterative time discretization scheme is adopted to restore global equilibrium of the beam. Integration of the inelastic rate equations is performed for each monitoring station with an efficient iterative process and stress resultants are obtained employing incremental strains.
v. The beam is supported by the most general nonlinear boundary conditions including elastic support or restrain, while its cross section is an arbitrarily monosymmetric one.

Numerical results are worked out to illustrate the method, demonstrate its efficiency and wherever possible its accuracy.

Statement of the problem

Let us consider a prismatic beam of length l (Fig. 1) of arbitrary constant cross–section having at least one axis of symmetry (z-axis), occupying the two dimensional multiply connected region Ω of the y,z plane bounded by the $\Gamma_j \left(j = 1,2,...,K \right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1 Cyz is the principal bending coordinate system through the cross section's centroid. The normal stress-strain relationship for the material is assumed to be elastic-plastic-strain hardening with initial modulus of elasticity and yield stress E and σ_{Y0}, respectively (Fig. 1). The beam is resting on nonlinear inelastic tensionless Winkler type foundation and thus the foundation reaction is expressed as

$$p_f = \begin{cases} k_w w & \text{if } p_f > 0 \\ 0 & \text{if } p_f \le 0 \end{cases} \tag{1}$$

where $k_w = k_w \left(w, w_y \right)$ is the Winkler nonlinear inelastic functions depending on the yielding displacement and the current one (Fig. 1).

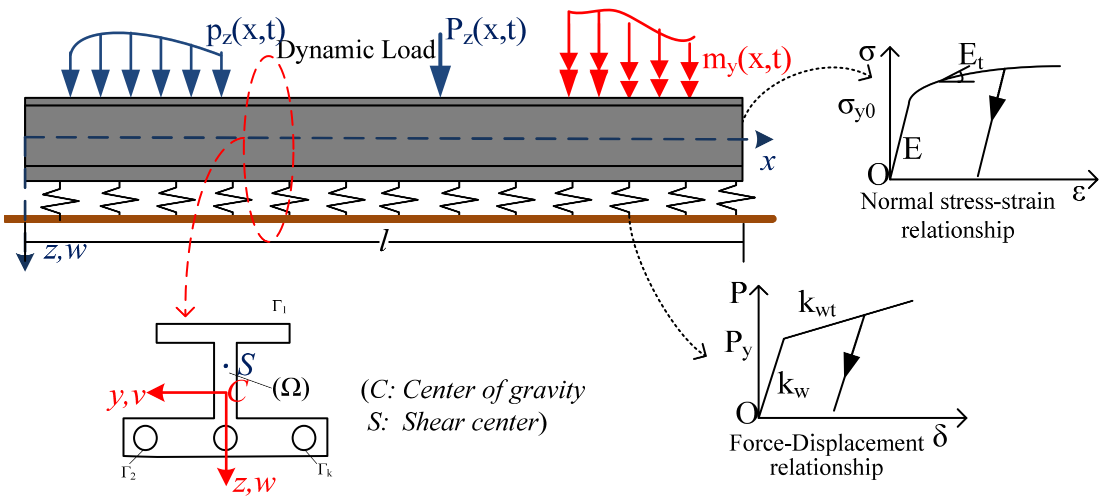


Fig 1: Prismatic beam resting on an inelastic foundation subjected to dynamic bending loading with an arbitrary cross-section having at least one axis of symmetry, occupying the two dimensional region Ω.

The beam is subjected to the combined action of arbitrarily distributed or concentrated time dependent transverse loading $p_z = p_z \left(x,t \right)$ and bending moment $m_y = m_y \left(x,t \right)$ acting in the z direction (Fig. 1). Under the action of the aforementioned loading, the displacement field of the beam is given as

$$\bar{u}(x,z,t) = u\left(x,t \right) + z\theta_y \left(x,t \right) \tag{2a}$$

$$\bar{w}\left(x,t \right) = w\left(x,t \right) \tag{2b}$$

where $\bar{u}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u\left(x,t \right)$, $w\left(x,t \right)$ are the corresponding components of the centroid C and $\theta_y \left(x,t \right)$ is the angle of rotation due to bending of the cross-section with respect to its centroid. Employing the strain-displacement relations considering small deflections and adopting the Euler-Bernoulli assumption the following strain components are obtained

$$\varepsilon_{xx} = -z\frac{d^2 w}{dx^2} \tag{3a}$$

$$\gamma_{xz} = 0 \Rightarrow \theta_y = -\frac{dw}{dx} \tag{3b}$$

Considering strains to be small, employing the Cauchy stress tensor and assuming an isotropic and homogeneous material without exhibiting any damage during its plastification, the normal stress rate is defined in terms of the corresponding strain one as

$$d\sigma_{xx} = E^* d\varepsilon_{xx}^{el} \tag{4}$$

where $d(\cdot)$ denotes infinitesimal incremental quantities over time (rates), the superscript el denotes the elastic part of the strain component and $E^* = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$. If the plane stress hypothesis is undertaken then $E^* = \frac{E}{1-\nu^2}$ holds , while E is frequently considered instead of E^* ($E^* \approx E$) in beam formulations. This last consideration has been followed throughout the paper, while any other reasonable expression of E^* could also be used without any difficulty in many beam formulations.

As long as the material remains elastic or elastic unloading occurs ($d\varepsilon_{xx} = d\varepsilon_{xx}^{el}$) the stress rate is given with respect to the strain one from eqn. (4). If plastic flow occurs then $d\varepsilon_{xx} = d\varepsilon_{xx}^{el} + d\varepsilon_{xx}^{pl}$, where the superscript pl denotes the plastic part of the strain component. The Von Mises yielding criterion is considered ignoring the influence of shear stresses and the yield condition is satisfied when the normal stress is equated with the yield stress of the material, that is

$$f = \sigma_{xx} - \sigma_Y\left(\varepsilon_{eq}^{pl}\right) = 0 \tag{5}$$

where σ_Y is the yield stress of the material and ε_{eq}^{pl} is the equivalent plastic strain, the rate of which is defined in [9] and is given as $d\varepsilon_{eq}^{pl} = d\lambda$ ($d\lambda$ is the proportionality factor). Moreover, the plastic modulus h is defined as $h = d\sigma_Y / d\varepsilon_{eq}^{pl}$ or $d\sigma_Y = hd\lambda$ and can be estimated from a tension test as $h = E_t E / (E - E_t)$. The stress rate is given with respect to the total strain one through eqn. (3) and the strain components as

$$d\sigma_{xx} = Ed\varepsilon_{xx} - Ed\varepsilon_{xx}^{pl} \tag{6}$$

Equations of global equilibrium

On the basis of Hamilton's principle, the variations of the Lagrangian equation defined as

$$\delta\int_{t_1}^{t_2}\left(U - K - W_{ext}\right)dt = 0 \tag{7}$$

and expressed as a function of the stress resultants acting on the cross section of the beam provide the governing equations and the boundary conditions of the beam. In eqn.(7) $\delta(\cdot)$ denotes variation of quantities, V is the volume and l is the length of the beam, while U, K, W_{ext} are the strain energy, the kinetic energy and the external load work, respectively given as

$$\delta U=\int_V\left(\sigma_{xx}\delta\varepsilon_{xx}\right)dV \qquad \delta K=\frac{1}{2}\int_V\rho\left(\delta\dot{w}^2\right)dV \qquad \delta W_{ext}=\int_l\left(p_z\delta w-\left(p_f\right)\delta w+m_y\delta\theta_y\right)dx \tag{8a,b,c}$$

The stress resultant corresponding to the internal bending moment of the beam is defined as

$$SM_y=\int_\Omega\sigma_{xx}zd\Omega \tag{9}$$

After substituting eqn. (9) into eqn. (7) and conducting some algebraic manipulations, the global equilibrium equations of the beam is obtained as

$$-\frac{d^2SM_y}{dx^2}+\rho A\ddot{w}+p_f\left(x\right)=p_z\left(x\right)+\frac{dm_y\left(x\right)}{dx} \tag{10}$$

along with its corresponding boundary conditions

$$\alpha_1\frac{dSM_y}{dx}+\alpha_2 w=\alpha_3 \qquad \beta_1 SM_y+\beta_2\frac{dw}{dx}=\beta_3 \tag{11a,b}$$

at the beam ends $x=0,l$, together with the initial conditions

$$w\left(x,0\right)=\bar{w}_0\left(x\right) \qquad \dot{w}\left(x,0\right)=\dot{\bar{w}}_0\left(x\right) \tag{12a,b}$$

where $\bar{w}_0\left(x\right)$ and $\dot{\bar{w}}_0\left(x\right)$ are prescribed functions, while α_i, β_i $(i=1,2,3)$ are functions specified at the beam ends. The boundary conditions (11) are the most general ones for the problem at hand, including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) may be derived from eqns. (11) by specifying appropriately the functions α_i and β_i (e.g. for a clamped edge it is $\alpha_2=\beta_2=1$, $\alpha_1=\alpha_3=\beta_1=\beta_3=0$).

Using the definition of eq. (9), elastoplastic constitutive equations at the beam level are derived by cross-sectional integration of eq. (6)

$$SM_y=-EI_y w''-SM_y^{pl} \tag{13}$$

where I_y is the moment of inertia with respect to the principle bending axis *y* and SM_y^{pl} is the plastic quantity defined as

$$SM_y^{pl}=E\int_\Omega\Delta\varepsilon^{pl}zd\Omega \tag{14}$$

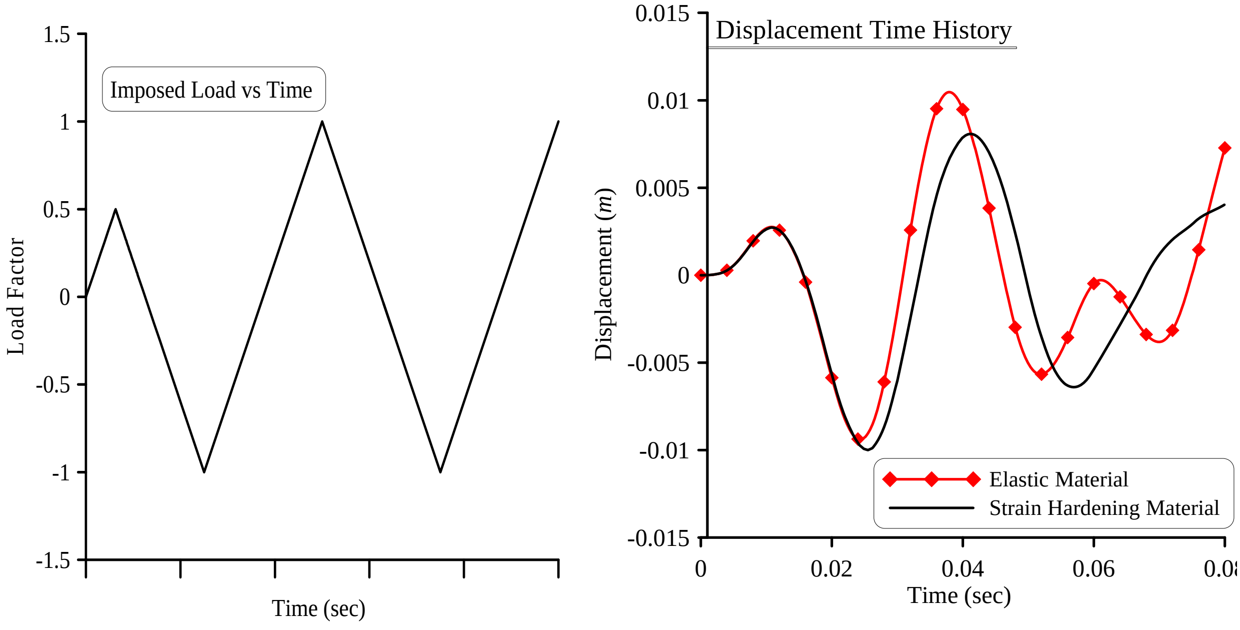
Integral Representations – Numerical Solution

According to the precedent analysis, the dynamic inelastic problem of an Euler-Bernoulli beams resting on resting on inelastic foundation, reduces to establishing the displacement component $w\left(x,t\right)$ having continuous derivatives up to the fourth order with respect to x and up to the second order with respect to *t*, satisfying the initial boundary value problem described by the governing differential equation (10) along the beam, the boundary conditions (eqs. (11)) and the initial conditions (eqs (12)) at the beam ends $x=0,l$. This initial boundary value problem is solved employing the BEM [8], as this is developed in [10] for the solution of a fourth order differential equation with constant coefficients, after some modifications.

Numerical Example

A rectangular cross section ($h = 0.60m$, $b = 0.30m$) pinned–fixed beam of length $l = 6.0m$ resting on a Winkler foundation with initial stiffness $k_w = 20MPa$ and yielding force $P_{wY} = 100kN / m$ has been studied. For the conducted analysis 20 linear longitudinal elements, 400 boundary elements, 72 quadrilateral cells and a 3×3 Gauss integration scheme for each cell, have been employed. The beam is subjected to a dynamic uniformly distributed loading acting at the first half of the beam's length and following the time function presented in Fig. 2(a). The beam's material is assumed to follow elastoplastic-strain hardening law with $E = 32318.4MPa$, $\sigma_{Y0} = 20MN / m^2$ and $E_t = 650MPa$.

In Fig. 2(b) the time history of the deflection $w(l/2,t)$ of the midpoint of the beam subjected to the aforementioned dynamic loading is presented, performing either elastic or inelastic analysis ignoring the foundation reaction. Moreover, in Table 1 the static deflection of the midpoint of the beam is presented for different load stages and material properties taking into account or ignoring the elastic-plastic Winkler foundation reaction, respectively.



(a) (b)

Fig. 2 Dynamic load vs. time function (a). Time history of the midpoint deflection $w(l/2,t)$ *of the beam ignoring the foundation reaction (b).*

Concluding Remarks

In this investigation a Boundary Element Method (BEM) is developed for the elastoplastic dynamic analysis of an Euler-Bernoulli beam of simply or multiply connected constant cross section having at least one axis of symmetry, resting on inelastic foundation. The main conclusions that can be drawn from this investigation are

a. The numerical technique presented in this investigation is well suited for computer aided analysis of prismatic beams of arbitrary simply or multiply connected cross section having at least one axis of symmetry, supported by the most general boundary conditions and subjected to the action of arbitrarily distributed or concentrated vertical loading.
b. The inelastic analysis and the soil nonlinearity are of paramount importance for the dynamic response of the beam-foundation system.
c. Accurate results are obtained using a relatively small number of nodal points across the longitudinal axis.

d. A small number of cells (fibers) is required in order to achieve satisfactory convergence.

e. The developed procedure retains most of the advantages of a BEM solution even though domain discretization is required.

Perfectly Plastic Winkler Foundation			
$p_z / w(l/2)$	Elastic – $E_t = E$	Perfectly Plastic – $E_t = 0$	Strain Hardening – E_t
350	0.495	0.499	0.512
420	0.594	0.801	0.780
480	0.679	-	2.186
Ignoring Foundation Reaction			
$p_z / w(l/2)$	Elastic – $E_t = E$	Perfectly Plastic – $E_t = 0$	Strain Hardening – E_t
150	0.344	0.344	0.345
290	0.666	1.685	0.952
325	0.746	-	2.176

Table 1: Midpoint deflection $w(l/2)$ (cm) of the beam for different types of beam and foundation material properties

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