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Paper 1

Cyclic Inelastic Response of Beam-Foundation Systems using the Boundary Element Method

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Keywords: inelastic analysis, cyclic loading, beam on foundation, inelastic Winkler model, distributed plasticity, boundary element method.

In this paper a boundary element method (BEM) is developed for the inelastic analysis of beams of arbitrarily shaped constant cross section having at least one axis of symmetry, resting on a nonlinear inelastic foundation. The beam is subjected to arbitrarily distributed or concentrated vertical cyclic loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modelled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative solution strategy is adopted to resolve both the plastic part of stress resultants and the foundation reaction along with an efficient iterative process to integrate the inelastic rate equations. The arising boundary value problem is solved employing the BEM.

Abstract

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1 Introduction

In engineering practice we often come across the analysis of beams on/in soil medium. The beam-foundation analysis is often required in piles, pile-columns and pile groups embedded in soil medium as well as in beam-columns and railway tracks resting on soil half space. These beam-foundation systems under the action of cyclic loading are usually leaded to the structural's element and/or soil yielding [1-4]. Moreover, design of beams and engineering structures based on elastic analysis are most likely to be extremely conservative not only due to significant difference between initial yield and full plastification in a cross section, but also due to the unaccounted for yet significant reserves of strength that are not mobilized in redundant members until after inelastic redistribution takes place. Thus, material nonlinearity is important for investigating the ultimate strength of a beam that resists bending loading, while distributed plasticity models are acknowledged in the literature [5-7] to capture more rigorously material nonlinearities than cross sectional stress resultant approaches [8] or lumped plasticity idealizations [9, 10].

Contrary to the large amount of research concerning the elastic analysis of beams on either linear or nonlinear elastic foundation [11-15], only few studies have taken into account the *inelastic* behavior of both the beam and the foundation. According to this, the beam stress-strain and the foundation load-displacement relations are assumed to follow nonlinear inelastic constitutive laws. Consequently, such beam-foundation models are not commonly used due to the complexity of the problem. Ayoub [16] presented an inelastic finite element formulation for that is capable of capturing the nonlinear behaviour of both the beam and the foundation. The element is derived from a two-field mixed formulation with independent approximation of forces and displacements and compared with the displacement based formulation. Lately, Mullapudi and Ayoub [17] expanded the research in inelastic analysis of beams resting on two-parameter foundation where the values for the parameters are derived through an iterative technique that is based on an assumption of plane strain for the soil medium.

In this paper, a Boundary Element Method (BEM) is developed for the inelastic analysis of beams of arbitrarily shaped constant cross section having at least one axis of symmetry, resting on nonlinear inelastic foundation. The beam is subjected to arbitrarily distributed or concentrated vertical cyclic loading along its length, while its edges are subjected to the most general boundary conditions. A displacement based formulation is developed and inelastic redistribution is modelled through a distributed plasticity model exploiting material constitutive laws and numerical integration over the cross sections. An incremental - iterative solution strategy is adopted to resolve both the plastic part of stress resultants and the foundation reaction along with an efficient iterative process to integrate the inelastic rate equations [18]. The arising boundary value problem is solved employing BEM [19]. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows

i. The formulation is a displacement based one taking into account inelastic redistribution along the beam axis by exploiting material constitutive laws and numerical integration over the cross sections (distributed plasticity approach).
ii. The cyclic response of the beam-foundation system is thoroughly examined and the influence of the material hardening is investigated.
iii. The inelasticity of the soil medium is taken into account, employing the nonlinear Winkler foundation model.
iv. The tensionless character of the foundation is also taken into consideration.
v. An incremental - iterative solution strategy is adopted to restore global equilibrium of the beam.
vi. The beam is supported by the most general nonlinear boundary conditions including elastic support or restrain, while its cross section is an arbitrarily one having at least one axis of symmetry (z-axis).
vii. To the authors' knowledge, a BEM approach has not yet been used for the solution of the aforementioned problem, while the developed procedure retains

most of the advantages of a BEM solution even though domain discretization is required.

A representative numerical application has been studied to demonstrate the efficiency and the accuracy of the developed method.

2 Statement of the Problem

2.1 Displacements, strains, stresses

Let us consider a prismatic beam of length l (Figure 1a) of arbitrary constant cross-section having at least one axis of symmetry (z-axis), occupying the two dimensional multiply connected region Ω of the y,z plane bounded by the $\Gamma_j\left(j=1,2,...,K\right)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Figure 1b Cyz is the principal bending coordinate system through the cross section's centroid. The normal stress-strain relationship for the material is assumed to be elastic-plastic-strain hardening with initial modulus of elasticity and yield stress E and σ_{Y0}, respectively. The beam is resting on nonlinear inelastic tensionless Winkler type foundation and thus the foundation reaction is expressed as

$$p_f=\begin{cases}k_w w & \text{if } p_f>0\\ 0 & \text{if } p_f\leq 0\end{cases} \tag{1}$$

where $k_w=k_w\left(w,w_y\right)$ is the Winkler nonlinear inelastic functions depending on the yielding displacement and the current one.

The beam is subjected to the combined action of arbitrarily distributed or concentrated cyclic transverse loading $p_z=p_z\left(x\right)$ and bending moment $m_y=m_y\left(x\right)$ acting in the x direction (Figure 1a). Under the action of the aforementioned loading, the displacement field of the beam is given as

$$\bar{u}(x,z)=u\left(x\right)+z\theta_y \tag{2a}$$

$$\bar{w}\left(x\right)=w\left(x\right) \tag{2b}$$

where $\bar{u}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Cyz system of axes; $u\left(x\right)$, $w\left(x\right)$ are the corresponding components of the centroid C and $\theta_y\left(x,t\right)$ is the angle of rotation due to bending of the cross-section with respect to its centroid. Employing the strain-displacement relations considering small deflections and adopting the Euler-Bernoulli assumption the following strain components are obtained

$$\varepsilon_{xx} = -z\frac{d^2w}{dx^2} \tag{3a}$$

$$\gamma_{xz} = 0 \Rightarrow \theta_y = -\frac{dw}{dx} \tag{3b}$$

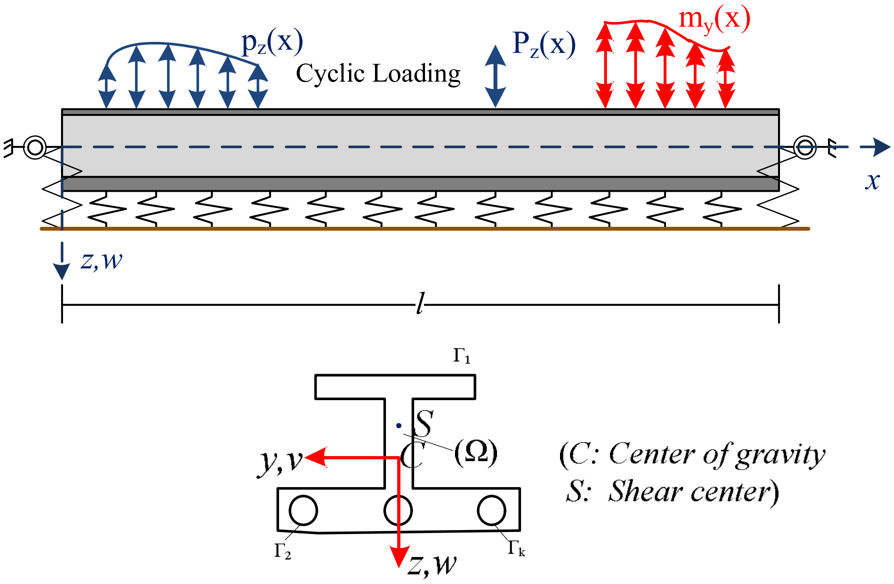

Figure 1: Prismatic beam resting on an inelastic foundation subjected to cyclic bending loading (a) with an arbitrary cross-section having at least one axis of symmetry, occupying the two dimensional region Ω (b)

Considering strains to be small, employing the Cauchy stress tensor and assuming an isotropic and homogeneous material without exhibiting any damage during its plastification, the normal stress rate is defined in terms of the corresponding strain one as

$$d\sigma_{xx} = E^* d\varepsilon_{xx}^{el} \tag{4}$$

where $d(\cdot)$ denotes infinitesimal incremental quantities over time (rates), the superscript el denotes the elastic part of the strain component and $E^* = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$. If the plane stress hypothesis is undertaken then $E^* = \frac{E}{1-\nu^2}$ holds, while E is frequently considered instead of E^* ($E^* \approx E$) in beam formulations. This last consideration has been followed throughout the paper, while any other reasonable expression of E^* could also be used without any difficulty in many beam formulations.

As long as the material remains elastic or elastic unloading occurs ($d\varepsilon_{xx} = d\varepsilon_{xx}^{el}$) the stress rate is given with respect to the strain one from eqn. (4). If plastic flow occurs then $d\varepsilon_{xx} = d\varepsilon_{xx}^{el} + d\varepsilon_{xx}^{pl}$, where the superscript pl denotes the plastic part of the strain component. A simplified Von Mises yielding criterion is considered ignoring the influence of shear stresses and the yield condition is satisfied when the normal stress is equated with the yield stress of the material, that is

$$f = \sigma_{xx} - \sigma_Y\left(\varepsilon_{eq}^{pl}\right) = 0 \tag{5}$$

where σ_Y is the yield stress of the material and ε_{eq}^{pl} is the equivalent plastic strain, the rate of which is defined in [20] and is given as $d\varepsilon_{eq}^{pl} = d\lambda$ ($d\lambda$ is the proportionality factor [20]). Moreover, the plastic modulus h is defined as $h = d\sigma_Y / d\varepsilon_{eq}^{pl}$ or $d\sigma_Y = hd\lambda$ and can be estimated from a tension test as $h = E_t E / (E - E_t)$ (Figure 2). The stress rate is given with respect to the total strain one through eqn. (3) and the strain components as

$$d\sigma_{xx} = Ed\varepsilon_{xx} - Ed\varepsilon_{xx}^{pl} \tag{6}$$

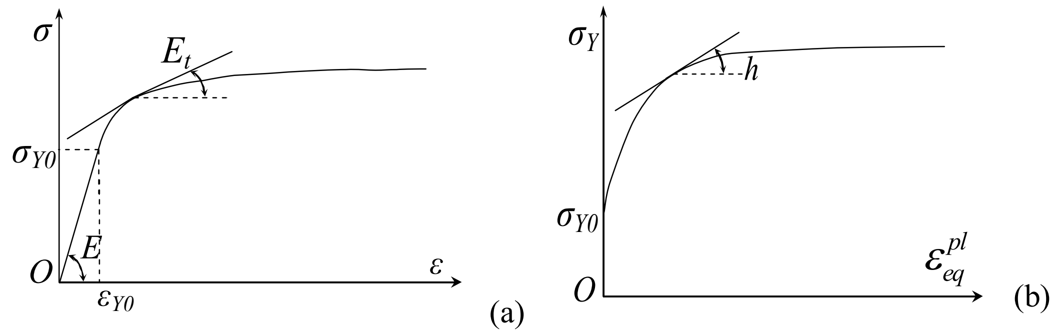

Figure 2: Normal stress - strain (a) and yield stress–equivalent plastic strain (b) relationships

2.2 Equations of global equilibrium

To establish global equilibrium equations, the principle of virtual work neglecting body forces is employed, that is

$$\int_V \left(\sigma_{xx}\delta\varepsilon_{xx}\right) dV = \int_l \left(p_z\delta w - m_y\delta w'\right) dx - \int_l \left(p_f\right) \delta w dx \tag{7}$$

where the integral quantities represent the strain energy, the external load and foundation reaction work while $\delta(\cdot)$ denotes virtual quantities, V is the volume and

l is the length of the beam. The stress resultant corresponding to the internal bending moment of the beam is defined as

$$SM_y = \int_{\Omega} \sigma_{xx} z d\Omega \tag{8}$$

After substituting eqn. (8) into eqn. (7) and conducting some algebraic manipulations, the global equilibrium equations of the beam is obtained as

$$-\frac{d^2 SM_y}{dx^2} + p_f(x) = p_z(x) + \frac{dm_y(x)}{dx} \tag{9}$$

along with its corresponding boundary conditions

$$\alpha_1 \frac{dSM_y}{dx} + \alpha_2 w = \alpha_3 \tag{10a}$$

$$\beta_1 SM_y + \beta_2 \frac{dw}{dx} = \beta_3 \tag{10b}$$

where α_i, β_i $(i = 1,2,3)$ are functions specified at the beam ends. The boundary conditions (10) are the most general ones for the problem at hand, including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) may be derived from eqns (10) by specifying appropriately the functions α_i and β_i (e.g. for a clamped edge it is $\alpha_2 = \beta_2 = 1$, $\alpha_1 = \alpha_3 = \beta_1 = \beta_3 = 0$).

Since an incremental - iterative approach is adopted for the problem at hand, the incremental version of eqns (9, 10) is firstly written down as

$$-\frac{d^2 \Delta SM_y}{dx^2} + \Delta p_f(x) = \Delta p_z(x) + \frac{d\Delta m_y(x)}{dx} \tag{11}$$

where $\Delta(\cdot)$ denotes incremental quantities (over time), while the incremental stress resultant is given by virtue of eqns (8) and (6) as

$$\Delta SM_y = -EI_y \Delta w'' - \Delta SM_y^{pl} \tag{12}$$

where I_y is the moment of inertia with respect to the principle bending axis y and SM_y^{pl} is the plastic quantity defined as

$$\Delta SM_y^{pl} = E \int_{\Omega} \Delta\varepsilon^{pl} z d\Omega \tag{13}$$

By substituting eqn. (12) in eqn. (11) and forming the incremental version of the boundary conditions (eqns. (10)), the following boundary value problem is obtained

$$EI_y \Delta w'''' + \Delta\left(p_f\right) = \Delta p_z\left(x\right) + \frac{d\Delta m_y\left(x\right)}{dx} - \frac{d^2 \Delta SM_y^{pl}}{dx^2} \quad \text{inside the beam} \tag{14}$$

$$\left.\begin{aligned} a_1 \frac{d\Delta SM_y}{dx} + a_2 \Delta w = \Delta a_3 \\ \beta_1 \Delta SM_y + \beta_2 \frac{d\Delta w}{dx} = \Delta\beta_3 \end{aligned}\right\} \text{at the beam ends } x = 0,l \tag{15-16}$$

By dropping the plastic quantities of the above equations, the boundary value problem of the examined problem under elastic conditions is formulated.

3 Integral Representations – Numerical Solution

According to the precedent analysis, the inelastic problem of beams resting on resting on nonlinear inelastic foundation, reduces to establishing the displacement component $\Delta w(x)$ having continuous derivatives up to the fourth order with respect to x and satisfying the boundary value problem described by the governing differential equation (14) along the beam and the boundary conditions (15-16) at the beam ends $x = 0,l$. This boundary value problem (eqns (14), (15-16)) is solved employing the BEM [19], as this is developed in [21] for the solution of a fourth order differential equation with constant coefficients, after some modifications.

3.1 Incremental - iterative solution algorithm

The initial stiffness method has been implemented in the present study, since (i) it requires exclusively BEM computations to obtain the stiffness matrix (which is computed only once and stored in the beginning of the algorithm) and (ii) no convergence difficulties have arise. Moreover, load control over the incremental steps is used and load stations are chosen according to the load history and convergence requirements. Incremental stress resultants are decomposed into elastic and plastic part and they are computed through an iterative procedure.

4 Numerical Example

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and a representative numerical application has been studied to demonstrate the efficiency and the accuracy of the developed method.

4.1 Example

A rectangular cross section ($h = 0.60m$, $b = 0.30m$) pinned–fixed beam of length $l = 6.0m$ resting on an elastic–plastic Winkler foundation with initial stiffness $k_w = 20MPa$ and yielding force $P_{wY} = 100kN/m$ has been studied. For the conducted analysis 20 linear longitudinal elements, 400 boundary elements, 72 quadrilateral cells and a 3×3 Gauss integration scheme for each cell, have been employed. The beam is subjected to a cyclic uniformly distributed loading acting at $0 \leq x \leq 3.0m$. Two material cases have been analyzed; namely an elastic-perfectly plastic with $E = 32318.4MPa$, $\sigma_{Y0} = 20MN/m^2$ and $E_t = 0$ and an elastoplastic-strain hardening with $E_t = 650MPa$.

In Figures 3, 4 the load–displacement curves at the midpoint $w(l/2)$ of the beam are presented for different types of material properties ignoring the foundation reaction or accounting for the elastic-plastic Winkler foundation, respectively. Moreover, in Table 1 the maximum beam deflection $w_{\max}$ is presented for different load stages and material properties as compared with those obtained from two finite element models (FEM) [22]; namely a 3–D solid one employing 2561 solid elements and 81 nonlinear springs and a 1–D model employing 120 beam and spring elements. The convergence between the proposed formulation and the solid simulation, as well as the inability of the beam FEM to capture accurately the systems response is observed. From these figures and table, the significant influence of the inelastic analysis to the beam-foundation response, as well as the reliability of the proposed method are verified.

5 Concluding Remarks

In this paper a BEM approach is developed for the inelastic analysis of beams of arbitrarily shaped constant cross section having at least one axis of symmetry, resting on tensionless inelastic foundation. The main conclusions that can be drawn from this investigation are

a. The numerical technique presented in this investigation is well suited for computer aided analysis of prismatic beams of arbitrary simply or multiply connected cross section having at least one axis of symmetry, supported by the most general boundary conditions and subjected to the action of arbitrarily distributed or concentrated vertical loading.
b. The inelastic analysis and the soil nonlinearity are of paramount importance for the cyclic response of the beam-foundation system.
c. Accurate results are obtained using a relatively small number of nodal points across the longitudinal axis.

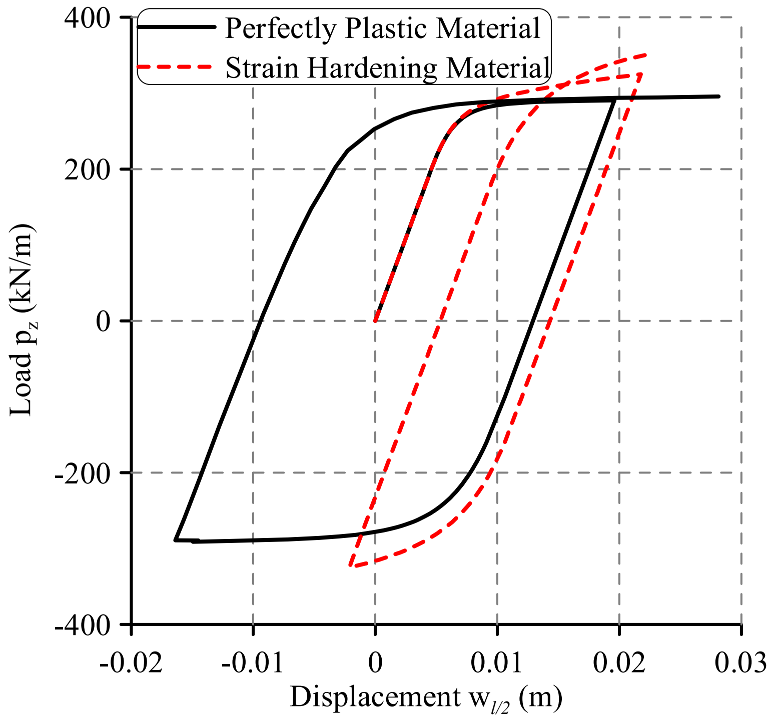

Figure 3: Load–displacement curve at the midpoint of the beam ignoring the foundation reaction

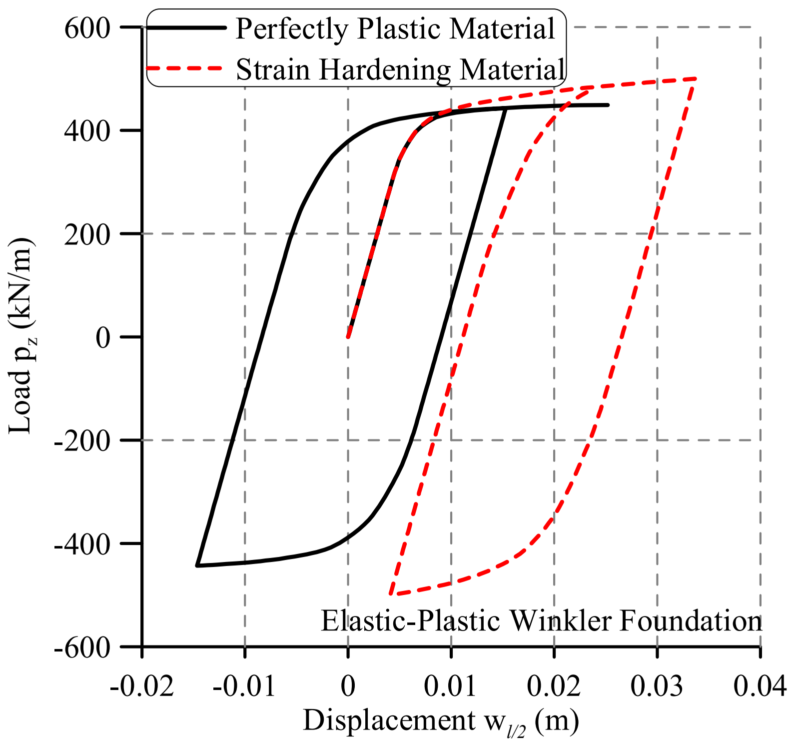

Figure 4: Load–displacement curve at the midpoint of the beam accounting for the elastic-plastic Winkler foundation

Elastic Winkler Foundation						
$p_z / w_{\max}$	Perfectly Plastic $E_t = 0$			Strain Hardening $E_t = 650 MN/m^2$		
	Present Study	FEM Solid Model [22]	FEM Beam Model [22]	Present Study	FEM Solid Model [22]	FEM Beam Model [22]
500	0.987	1.002	1.100	0.980	0.984	1.041
550	1.202	1.213	-	1.158	1.170	1.252
600	1.438	1.468	-	1.364	1.384	1.483
Perfectly Plastic Winkler Foundation						
$p_z / w_{\max}$	Perfectly Plastic $E_t = 0$			Strain Hardening $E_t = 650 MN/m^2$		
	Present Study	FEM Solid Model [22]	FEM Beam Model [22]	Present Study	FEM Solid Model [22]	FEM Beam Model [22]
350	0.567	0.589	0.576	0.566	0.585	0.586
400	0.767	0.780	0.758	0.756	0.769	0.811
440	1.657	1.659	-	1.199	1.215	2.128
Hardening ($k_{wt} = 1.0 MPa$) Winkler Foundation						
$p_z / w_{\max}$	Perfectly Plastic $E_t = 0$			Strain Hardening $E_t = 650 MN/m^2$		
	Present Study	FEM Solid Model [22]	FEM Beam Model [22]	Present Study	FEM Solid Model [22]	FEM Beam Model [22]
400	0.750	0.773	0.810	0.743	0.766	0.789
450	1.663	1.632	-	1.254	1.285	1.938
500	5.689	5.651	-	2.618	2.678	3.876

Table 1: Maximum deflection $w_{\max}$ (cm) of the beam for different types of beam and foundation material properties

d. A small number of cells (fibers) is required in order to achieve satisfactory convergence.
e. The developed procedure retains most of the advantages of a BEM solution even though domain discretization is required.

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